

# Closure of the set of diffusion functionals with respect to the Mosco-convergence.

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## Abstract

We characterize the functionals which are Mosco-limits, in the  $L^2(\Omega)$  topology, of some sequence of functionals of the kind

$$F_n(u) := \int_{\Omega} \alpha_n(x) |\nabla u(x)|^2 dx ,$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  ( $N \geq 3$ ). It is known that this family of functionals is included in the closed set of Dirichlet forms. Here, we prove that the set of Dirichlet forms is actually the closure of the set of diffusion functionals. A crucial step is the explicit construction of a composite material whose effective energy contains a very simple non-local interaction.

**keywords** : Homogenization, Mosco-convergence,  $\Gamma$ -convergence, Dirichlet forms, composite materials.

## 1 Introduction

Many physical situations are described by the minimization of a diffusion functional of the kind

$$F_{\alpha}(u) := \int_{\Omega} \alpha(x) |\nabla u(x)|^2 dx , \tag{1.1}$$

where the diffusion coefficient of the material  $\alpha$  may vary from place to place. An important area of investigation is the asymptotic analysis of such functionals when  $\alpha$  depends on some small parameter. This is the case for instance when studying a diffusion process in a porous medium or in a composite one in which the small parameter is characteristic of the length-scale of the in-homogeneities. Then one has to consider a family  $(F_{\alpha_n})$  of diffusion functionals and to search for its limit  $F$ . The functional  $F$  describes the effective properties of the homogenized material. These properties can sometimes be described by a homogenized diffusion coefficient  $\alpha^{\text{hom}}$  but the homogenized material can also be non

isotropic : it is then described by a diffusion matrix  $A(x)$  and the limit functional takes the form :

$$\int_{\Omega} \nabla u(x) \cdot A(x) \cdot \nabla u(x) dx . \quad (1.2)$$

It has been proved [17] that the limit of functionals (1.1) takes the form (1.2) when the sequence of diffusion coefficients  $(\alpha_n)$  and their inverses  $(\alpha_n^{-1})$  are bounded by a fixed real  $M$ . This is still true under weaker assumptions [4] but not in the general case. Examples have been given (cf. [4], [2], [3] and [16]) in which non local interactions arise at the limit. These interactions are represented by a non-negative measure  $\mu$  on  $\Omega \times \Omega$  and the limit functional  $F$  contains the *non-local term* or *jumping term* :

$$\int_{\Omega \times \Omega} (u(x) - u(y))^2 \mu(dx, dy). \quad (1.3)$$

Other examples [18] have been given in which the limit functional contains a so-called *killing term* of the form

$$\int_{\Omega} (u(x))^2 \nu(dx), \quad (1.4)$$

where  $\nu$  is a non-negative measure on  $\Omega$ .

A natural question is to identify the different functionals which can be obtained as the asymptotic limits of diffusion problems. It is known [16] that any limit functional has to be a Dirichlet form, that is, in the regular case, a sum of terms of kind (1.2), (1.3) and (1.4). Here we consider the following inverse problem : is any given Dirichlet form the limit of a sequence of functionals of kind (1.1) ? Very recently Briane and Tchou [6] gave a partial positive answer to this question. They proved that any non-local term (1.3), in which the measure  $\mu$  has the particular form  $1_E(x)1_E(y) dx dy$ , can be reached. Our main results state that any Dirichlet form can actually be reached :

Let us call *objective* those functionals which vanish for constant fields  $u = c$ , or in an equivalent way, which are invariant when adding a constant :

$$\forall c \in \mathbb{R}, \quad F(u + c) = F(u). \quad (1.5)$$

We must consider two fundamental different situations :

i) if no Dirichlet condition is imposed in the initial diffusion problem, then the functionals (1.1) we have to consider are objective ones and any limit inherits this property. We focus on this situation in the first part of the paper where we prove that any objective Dirichlet form can be reached. In particular any non-local interaction of type (1.3) is the limit of some sequence of diffusion functionals. At the opposite, as they are non-objective quantities, killing terms cannot be obtained.

ii) if a Dirichlet condition is imposed in the initial diffusion problem, we prove (in section 7) that all possible killing terms can also be obtained.

Let us be more precise. The considered functionals are defined on  $L^2(\Omega)$  where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  ( $N \geq 3$ ). For sake of simplicity we assume that  $\Omega$  is the unit cube. We study the Mosco-limit of these functionals for the  $L^2(\Omega)$  topology.

Our main result (Theorem 1) states that the set of all objective Dirichlet forms coincides with the closure of the set of all diffusion functionals of type (1.1). This result is stated in section 2 where we fix our notations. The proof is obtained by the construction of a sequence of diffusion functionals  $(F_{\alpha_n})$  converging to a given objective Dirichlet form  $F$ . This is achieved in several steps. First, using the Yosida-Deny regularization procedure, we restrict our study to the case where  $F$  is a continuous functional on  $L^2(\Omega)$  (Theorem 3), then  $F$  takes the form (1.3). In a second step, discretizing the measure  $\mu$  which appears in (1.3), we approximate  $F$  by a finite combination of *elementary non-local interactions* (Theorem 4). Here we call *elementary non-local interaction* a term of kind (1.3) where  $\mu$  takes the particular form

$$\mu(dx, dy) := \delta_{x+w}(dy) \mathbf{1}_{\Omega}(x+w) f(x) dx , \quad (1.6)$$

$w$  being a given vector of  $\mathbb{R}^N$  and  $f$  a non-negative function in  $L^\infty(\Omega)$ . In this particular case, the non-local term (1.3) takes the form

$$\int_{\Omega} (u(x) - u(x+w))^2 \mathbf{1}_{\Omega}(x+w) f(x) dx . \quad (1.7)$$

Note that an elementary non-local interaction has direction and range fixed by  $w$ . In a third crucial step, we exhibit an explicit composite diffusive material whose effective properties contain the prescribed elementary interaction (1.6) (Theorem 5). An induction argument allows us to extend this result to a finite combination of such elementary interactions (Theorem 6).

In the last section (section 7), we extend the previous results to the family of diffusion functionals subjected to a Dirichlet condition. We prove (Theorem 2) that the closure of this family coincides with the set of all Dirichlet forms. The point is that any killing term (1.4) can be seen as the result of a non-local interaction between  $\Omega$  and the set where the Dirichlet condition is imposed. All these results were partially announced in [7].

Our results are stated when the dimension  $N$  of the ambient space is greater than or equal to three. The reader could notice that, the higher is the dimension, the easier are the proofs of our results. That is why our constructions, although valid for  $N \geq 3$ , are particularly adapted to the three-dimensional case. Our results are not valid in the one or two-dimensional cases : indeed, the crucial construction of a composite diffusive material converging to a non-local one cannot be performed in these cases. The point is that the topologies of  $\mathbb{R}$  or  $\mathbb{R}^2$  do not allow to introduce a set connecting two distant points without modifying drastically the diffusion process of the remaining part. To our knowledge, the characterization of the closure of the diffusion functionals is still an open problem in these cases.

The density results proved in this paper cannot be easily transposed to the case of elasticity functionals. Indeed, in this vector case, the theory of Dirichlet forms cannot be applied and the effective properties of a linear elastic composite material can fundamentally differ from those described by terms analogous with (1.2), (1.3) or (1.4). It has been proved in [19] that higher-order gradient terms can be present. In a forthcoming paper [8], using a similar approach as in the present study, we will prove that the Mosco-closure of the set of linear elasticity functionals coincides with the set of all non-negative, quadratic and lower semicontinuous functionals.

## 2 Main density results.

### 2.1 Notations and definitions :

Let  $\Omega := (0, 1)^N$  be the unit cube of  $\mathbb{R}^N$  ( $N \geq 3$ ). We denote  $L^2(\Omega)$  the usual Lebesgue space endowed with the norm  $\|u\|_{L^2(\Omega)} := (\int_{\Omega} |u(x)|^2 dx)^{1/2}$ .

Let  $\mathcal{B} := (0, 1)^{N-1} \times \{0\}$  denote a face of the cube  $\Omega$ . We denote  $H_{\mathcal{B}}^1(\Omega) := \{u \in H^1(\Omega), u = 0 \text{ on } \mathcal{B}\}$  where  $H^1(\Omega)$  is the usual Sobolev space, endowed with its standard norm.

We will also denote  $L^\infty(\Omega)$  the set of all essentially bounded Lebesgue measurable functions endowed with the usual norm  $\|u\|_{L^\infty(\Omega)} := \inf\{K; |u(x)| \leq K \text{ for a.e. } x \text{ in } \Omega\}$  and  $L_{++}^\infty(\Omega)$  the subset :

$$L_{++}^\infty(\Omega) := \{\varphi \in L^\infty(\Omega), 1/\varphi \in L^\infty(\Omega), \varphi \geq 0\}.$$

We denote  $|\mathcal{D}|$  the Lebesgue measure of any Borel set  $\mathcal{D}$  and  $f_{\mathcal{D}} u := |\mathcal{D}|^{-1} \int_{\mathcal{D}} u dx$  the mean value of any function  $u \in L^1(\mathcal{D})$ .

For any positive integer  $p$ , we denote  $\omega_p$  the unit ball of  $\mathbb{R}^p$  and  $|\omega_p|$  its volume.

### 2.2 Dirichlet forms

The functionals  $F$  we consider in this paper are defined on  $L^2(\Omega)$  and take value in  $\mathbb{R} \cup \{+\infty\}$ . They are proper non-negative quadratic functionals, i.e. there exist positive semidefinite bilinear forms  $B$  on  $D(F) := \{u \in L^2(\Omega) : F(u) < +\infty\}$  such that  $F(u) = B(u, u)$  for every  $u$  in  $D(F)$ . By proposition 11.9 of [11], these functionals are characterized by the fact that, for any  $u$  and  $v$  in  $L^2(\Omega)$  and any  $t \geq 0$ ,

$$F(u) \geq 0, \quad F(tu) \leq t^2 F(u), \quad F(u+v) + F(u-v) \leq 2F(u) + 2F(v). \quad (2.1)$$

They are said to be *lower semicontinuous* if they satisfy, for any  $u \in L^2(\Omega)$  and any sequence  $(u_n)$  converging to  $u$  :

$$\liminf_{n \rightarrow \infty} F(u_n) \geq F(u). \quad (2.2)$$

Such functionals are said to be *Markovian* if they satisfy for any  $u \in L^2(\Omega)$  :

$$F(\bar{u}) \leq F(u), \quad (2.3)$$

where  $\bar{u}$  denotes the truncated function  $\bar{u} := \sup(0, \inf(1, u))$ .

**Dirichlet forms :** A proper functional which is non-negative, quadratic, Markovian and lower semicontinuous is called a *Dirichlet form* ( cf. [14]). We denote  $\mathfrak{D}$  the set of all Dirichlet forms on  $L^2(\Omega)$ . They are characterized by (2.1), (2.3) and (2.2).

**Regular Dirichlet forms :** Let  $D(F)$  denote the domain of  $F$  :  $D(F) := \{u \in L^2(\Omega), F(u) < +\infty\}$  and  $C_0(\Omega)$  (or  $C_0^1(\Omega)$ ) the set of continuous (resp. continuously differentiable) functions with compact support in  $\Omega$ . If there exists a subset of  $D(F) \cap C_0(\Omega)$  dense in  $C_0(\Omega)$  for the uniform norm and in  $D(F)$  for the norm  $\sqrt{\|u\|_{L^2(\Omega)}^2 + F(u)}$ , then  $F$

is said to be a *regular Dirichlet form*. We denote  $\mathfrak{D}_r$  this important subclass of Dirichlet forms. The Deny-Beurling formula [5] states that any regular Dirichlet form admits on  $C_0^1(\Omega)$  the following representation (in which  $\nu$  and  $\mu$  are non-negative Radon measures respectively on  $\Omega$  and  $\Omega \times \Omega$ , while  $\eta$  is a Radon measure on  $\Omega$  taking values in the set of non-negative symmetric matrices) :

$$F(u) = \sum_{i,j=1}^N \int_{\Omega} \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \eta_{ij}(dx) + \int_{\Omega} (u(x))^2 \nu(dx) + \int_{\Omega \times \Omega} (u(x) - u(y))^2 \mu(dx, dy). \quad (2.4)$$

It is then the sum of three terms of type (1.2) (with  $A(x)dx$  replaced by  $\eta(dx)$ ), (1.4) and (1.3).

In the representation formula (2.4), the part of the measure  $\mu$  supported by the diagonal  $\Delta := \{(x, x), x \in \Omega\}$  does not play any role : then we assume

$$\mu(\Delta) = 0. \quad (2.5)$$

In the same way, note that only the symmetric part  $\mu^{sym}$  of the measure  $\mu$ , defined by

$$\mu^{sym}(A \times B) = \frac{1}{2} (\mu(A \times B) + \mu(B \times A)) \quad (2.6)$$

plays a role in (2.4). However we do not assume that  $\mu$  is symmetric. This allows us to use a simpler notation in the sequel.

Note that a functional  $F$  defined by (2.4) is not always a Dirichlet form : it may be not lower semicontinuous. However, if the measures  $\nu$  and  $\mu(dx, \Omega)$  do not charge polar sets (i.e. sets with vanishing  $H^1$ -capacity [13]) and if  $\alpha(x) \in L_{++}^{\infty}(\Omega)$  the particular functional

$$F_{\alpha, \nu, \mu}(u) := \begin{cases} \int_{\Omega} \alpha(x) |\nabla u(x)|^2 dx + \int_{\Omega} (u(x))^2 \nu(dx) \\ \quad + \int_{\Omega \times \Omega} (u(x) - u(y))^2 \mu(dx, dy), & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.7)$$

can be proved (cf. [9], [10]) to be lower semicontinuous. Then it is a Dirichlet form.

**Continuous Dirichlet forms :** We denote  $\mathfrak{D}_c$  the subset of all Dirichlet forms  $F$ , with  $D(F) = L^2(\Omega)$ , which are continuous in the strong topology of  $L^2(\Omega)$ . These functionals play an important role in our proofs. They are regular and, in their representation (2.4), the measures  $\eta_{ij}$  vanish and the measures  $\nu(dx)$ ,  $\mu^{sym}(dx, \Omega)$  are absolutely continuous with respect to  $dx$  with densities in  $L^{\infty}(\Omega)$ . Here  $\mu^{sym}(dx, \Omega)$  represents the projection of the symmetric part of  $\mu$ .

Let us sketch the proof of this characterization (for details, refer to [9]) : since  $F$  is continuous, there exists a constant  $C$  such that, for any  $u$  in  $L^2(\Omega)$ ,

$$F(u) \leq C \|u\|_{L^2(\Omega)}^2. \quad (2.8)$$

First, let us apply (2.8) to functions  $u$ , of class  $C^1$ , depending only on one component of  $x$  (say  $u(x) = f(x_i)$ ). Using Jensen inequality we get that the linear functional  $f \rightarrow$

$\int_0^1 f'(x_i) \eta_{ii}(dx_i)$  is continuous for the strong topology of  $L^2(0, 1)$  (here  $\eta_{ii}(dx_i)$  denotes the projection of  $\eta_{ii}(dx)$ ). Hence  $\eta_{ii}(dx_i)$  takes the form  $\eta_{ii}(dx_i) = g(x_i) dx_i$  with a continuous density  $g$ . If  $g \neq 0$ , considering functions  $f$  which vanish out of an interval  $I$  where  $g(x_i) > \epsilon > 0$ , we get

$$\int_I f'(x_i)^2 dx_i \leq C\epsilon^{-1} \int_I f(x_i)^2 dx_i.$$

Clearly this inequality cannot hold for every  $f$ . Then  $g$  has to be identically null. Therefore  $\eta_{ii}$ ,  $\text{Tr}(\eta)$  and then  $\eta$  are null.

Now, for any open set  $A \subset \Omega$  and any compact subset  $K \subset A$ , there exists a function  $u$  of class  $C^1$ , such that  $\mathbf{1}_K \leq u \leq \mathbf{1}_A$ . Inequality (2.8) implies  $\nu(K) \leq C|A|$ . This being true for any  $K \subset A$ , we have  $\nu(A) \leq C|A|$ . This is extended to any Borel set by approximation which proves the result for  $\nu$ . Using the same function  $u$ , we get also  $2\mu^{sym}(K \times A^c) \leq C|A|$  and then  $2\mu^{sym}(A \times A^c) \leq C|A|$ . Let us introduce the following neighborhood  $\Delta_n$  of the diagonal  $\Delta$  :  $\Delta_n := \{(x, y) \in \Omega \times \Omega : \|x - y\| > 1/n\}$  and  $\mu_n$  the restriction of  $\mu^{sym}$  to  $\Delta_n$ . When the diameter of  $A$  is small enough (lower than  $1/n$ ), we have

$$\mu_n(A \times \Omega) = \mu_n(A \times A^c) \leq \mu^{sym}(A \times A^c) \leq \frac{C}{2}|A|.$$

This is enough to prove that  $\mu_n(dx, \Omega)$  is absolutely continuous with respect to  $dx$  with a density bounded by  $C/2$ . Letting  $n$  tend to infinity and using the fact that  $\mu(\Delta) = 0$ , we obtain the same property for the measure  $\mu^{sym}(dx, \Omega)$ .

Conversely, if  $\nu$  and  $\mu$  are Radon measures on  $\Omega$  and  $\Omega \times \Omega$  such that  $\nu(dx)$  and  $\mu(dx, \Omega)$  are absolutely continuous with respect to  $dx$  with densities in  $L^\infty(\Omega)$ , then it is easy to see that the functional

$$F_{0,\nu,\mu}(u) := \int_\Omega (u(x))^2 \nu(dx) + \int_{\Omega \times \Omega} (u(x) - u(y))^2 \mu(dx, dy), \quad u \in L^2(\Omega), \quad (2.9)$$

belongs to  $\mathcal{D}_c$ . □

Possible measures  $\mu$ , are those which correspond to interactions with fixed direction and range. We call *elementary interactions* these measures and denote their set  $\mathcal{E}$  :

$$\mathcal{E} := \{\delta_{x+w}(dy) f(x) \mathbf{1}_\Omega(x+w) dx, \ w \in \Omega_2, \ f \in L^\infty(\Omega), \ f \geq 0\}, \quad (2.10)$$

where  $\Omega_2$  represents the set of vectors in  $\mathbb{R}^N$  with dyadic components

$$\Omega_2 := \{w \in \mathbb{R}^N, \ \exists p \in \mathbb{N} : 2^p w \in \mathbb{N}^N\}.$$

This restriction to vectors with dyadic components is purely technical. Its interest appears in the proof of theorem 5.

Finite combinations of such measures are also possible. We call *atomic interactions* these measures and denote their set  $\mathcal{A}$  :

$$\mathcal{A} := \left\{ \sum_{i=1}^n \delta_{x+w_i}(dy) f_i(x) \mathbf{1}_\Omega(x+w_i) dx, \ n \in \mathbb{N}, \ w_i \in \Omega_2, \ f_i \in L^\infty(\Omega), \ f_i \geq 0 \right\}. \quad (2.11)$$

Note that, as no confusion can arise, we will also call *elementary interaction* or *atomic interaction* the functional  $F_{0,0,\mu}$  when  $\mu$  belongs to  $\mathcal{E}$  or to  $\mathcal{A}$ .

**Objective Dirichlet forms :** We will also use the subset  $\mathfrak{D}_i$  of objective Dirichlet forms characterized by the property :

$$\forall c \in \mathbb{R}, \quad F(c) = 0 . \quad (2.12)$$

This property is clearly equivalent to (1.5). Indeed the quantity  $F(u+c) - F(u) - F(c)$  which is linear in  $c$  and lower-bounded by  $-F(u)$  has to vanish.

The sets of regular or continuous objective Dirichlet forms are denoted respectively  $\mathfrak{D}_{ri}$  and  $\mathfrak{D}_{ci}$ . Their characterization is obvious : the killing measure  $\nu$  has to vanish.

**Diffusion Dirichlet forms :** We call  $\mathfrak{D}_d$  the subset of isotropic diffusion functionals with non-degenerated and bounded diffusion coefficient, i.e. the set

$$\mathfrak{D}_d := \{F_{\alpha,0,0} ; \alpha \in L_{++}^\infty(\Omega)\} . \quad (2.13)$$

Note that there is no uniform bounds for the diffusion coefficient in the definition of  $\mathfrak{D}_d$  : very low or very high diffusion coefficients are admissible. Clearly  $\mathfrak{D}_d \subset \mathfrak{D}_i$ . The fact that  $\mathfrak{D}_d$  is a relatively “small” subset of diffusion forms does not weaken our results : the smaller  $\mathfrak{D}_d$ , the stronger the density result.

We will also consider the set  $\mathfrak{D}_0$  of diffusion functionals submitted to a Dirichlet condition on  $\mathcal{B}$  :

$$\mathfrak{D}_0 := \{F_{\alpha,0,0}^0 ; \alpha \in L_{++}^\infty(\Omega)\} \quad (2.14)$$

where

$$F_{\alpha,0,0}^0(u) := \begin{cases} \int_{\Omega} \alpha(x) |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \mathcal{B}, \\ +\infty & \text{otherwise.} \end{cases} \quad (2.15)$$

## 2.3 Mosco-convergence

The Mosco-convergence theory, introduced by U. Mosco [15], has been recognized as an appropriate framework to study the limit of convex variational problems. We refer to [1] for a detailed description of this theory. Let us recall the following characterization of the Mosco-convergence of a sequence of convex functionals :

**Definition 1** *A sequence of convex functionals  $(F_n)$  Mosco-converges to a functional  $F$  in the  $L^2(\Omega)$  topology, if and only if it satisfies the two following properties :*

*i) Lower-bound inequality : For any sequence  $(u_n)$  converging weakly to some  $u$  in  $L^2(\Omega)$ , the following lower-bound inequality holds :*

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F(u) .$$

ii) Upper-bound inequality : For every  $u$  in  $L^2(\Omega)$ , there exists an approximating sequence  $(u_n)$  converging to  $u$  strongly in  $L^2(\Omega)$  such that

$$\limsup_{n \rightarrow \infty} F_n(u_n) \leq F(u).$$

Then we write  $F_n \xrightarrow{L^2-M} F$ .

**Remark 1** It is clear that a Mosco-convergence result is proved even if one consider in (i) only sequences  $(u_n)$  with bounded “energy” (i.e. such that  $F_n(u_n) < M < +\infty$ ) and in (ii) only functions  $u$  such that  $F(u) < +\infty$ .

**Remark 2** A Mosco-convergence result established for the  $L^2(\Omega)$  topology is also established for the  $H_B^1(\Omega)$  topology if, for any  $u \in H_B^1(\Omega)$ , one can impose to the approximating sequence  $(u_n)$  in (ii) to converge to  $u$  strongly in  $H_B^1(\Omega)$ . In that case we say that  $F_n$   $\tau$ -converges to  $F$  and we write  $F_n \xrightarrow{\tau} F$ .

**Remark 3** If  $G$  is convex and continuous for the strong convergence of  $L^2(\Omega)$ , then

$$F_n \xrightarrow{L^2-M} F \implies F_n + G \xrightarrow{L^2-M} F + G, \quad (2.16)$$

$$F_n \xrightarrow{\tau} F \implies F_n + G \xrightarrow{\tau} F + G. \quad (2.17)$$

Indeed, if  $G$  is convex and continuous for the strong convergence of  $L^2(\Omega)$ , it is lower semicontinuous for the weak convergence of  $L^2(\Omega)$ . Then (2.16) follows immediately from Definition 1. Assertion (2.17) is due to the fact that  $G$  is also continuous for the strong convergence of  $H_B^1(\Omega)$ .

**Remark 4** For all  $c > 0$ , we have

$$F_n \xrightarrow{\tau} F \implies F_n + cF_{1,0,0}^0 \xrightarrow{\tau} F + cF_{1,0,0}^0. \quad (2.18)$$

Indeed, for any  $c > 0$  the functional  $cF_{1,0,0}^0$  is convex and lower semicontinuous for the strong convergence of  $L^2(\Omega)$ . Hence it is lower semicontinuous for the weak convergence of  $L^2(\Omega)$ . The lower-bound inequalities are assured. Consider now  $u$  such that  $F(u) + cF_{1,0,0}^0(u) < +\infty$ . Then  $u$  belongs to  $H_B^1(\Omega)$  and, as  $F_n$   $\tau$ -converges to  $F$ , there exists a sequence  $(u_n)$  converging to  $u$  strongly in  $H_B^1(\Omega)$  such that  $\limsup F_n(u_n) \leq F(u)$ . As  $cF_{1,0,0}^0$  is continuous for the strong topology of  $H_B^1(\Omega)$ , we have also  $\limsup F_n(u_n) + cF_{1,0,0}^0(u_n) \leq F(u) + cF_{1,0,0}^0(u)$ .

**Remark 5** Let  $(F_n)$  be a sequence which Mosco-converges to  $F$  for the  $L^2(\Omega)$ -topology (respectively, which  $\tau$ -converges to  $F$ ). Assume that, for any  $n$ , there exist a sequence  $(F_{n,m})$  which converges to  $F_n$  for the same topology as  $m$  tends to infinity. Then there exists a sequence of integers  $(m_n)$  such that the “diagonal” sequence  $(F_{n,m_n})$  Mosco-converges to  $F$  as  $n$  tends to infinity for the  $L^2(\Omega)$ -topology (resp., such that  $(F_{n,m_n})$   $\tau$ -converges to  $F$ ).

It is proved in [1] (section 3.5) that the topology of Mosco-convergence is metrizable on the set of proper lower semicontinuous convex functionals. Then, there exist a metric  $d_1$  and a pseudo-metric  $d_2$  on  $\mathfrak{D}$ , such that

$$F_n \xrightarrow{L^2-M} F \iff d_1(F_n, F) \rightarrow 0 \quad \text{and} \quad F_n \xrightarrow{H_B^1-M} F \iff d_2(F_n, F) \rightarrow 0.$$

Hence, the topology  $\tau$  associated to both convergence, is also metrizable :

$$F_n \xrightarrow{\tau} F \iff (d_1 + d_2)(F_n, F) \rightarrow 0.$$

The assertions of Remark 5 are nothing but the well known diagonalization property in metric spaces.

**Definition 2** *Let  $\mathfrak{U}$  be a subset of  $\mathfrak{D}$ , We call Mosco-closure of  $\mathfrak{U}$  and denote  $\overline{\mathfrak{U}}$  the set of all possible Mosco-limits for the  $L^2(\Omega)$  topology of all sequences in  $\mathfrak{U}$ .*

Note that, owing to Remark 5, we have  $\overline{\overline{\mathfrak{U}}} = \overline{\mathfrak{U}}$ .

**Remark 6** *The set of all Dirichlet forms and the set of all objective Dirichlet forms are closed :*

$$\overline{\mathfrak{D}} = \mathfrak{D} \quad \text{and} \quad \overline{\mathfrak{D}_i} = \mathfrak{D}_i . \tag{2.19}$$

Indeed, it is easily checked that any Mosco-limit is lower semicontinuous and that properties (2.1), (2.3) pass to the limit by Mosco-convergence. Then the Mosco-limit of any sequence of Dirichlet forms has to be a Dirichlet form. In the same way property (1.5) passes to the limit by Mosco-convergence : the limit of any sequence of objective Dirichlet forms is objective.

**Remark 7** *Mosco-convergence in the  $L^2(\Omega)$  topology is clearly a stronger notion than  $\Gamma$ -convergence for the strong topology of  $L^2(\Omega)$  (refer to [11] for definition and properties of  $\Gamma$ -convergence). Then the Mosco-closure of a set  $\mathfrak{U}$  is contained in its  $\Gamma$ -closure, i.e. in the set of all  $\Gamma$ -limits of all sequences in  $\mathfrak{U}$ . However, the closure results (2.19) remain true even if one uses the  $\Gamma$ -convergence in the strong topology of  $L^2(\Omega)$  [16]. Therefore all our density results can be interpreted in terms of  $\Gamma$ -convergence for the strong topology of  $L^2(\Omega)$ .*

## 2.4 Main results

This paper is devoted to the proof of the two following density results :

**Theorem 1** *The Mosco-closure in the  $L^2(\Omega)$ -topology of the set  $\mathfrak{D}_d$  of diffusion functionals coincides with the set of objective Dirichlet forms :  $\overline{\mathfrak{D}_d} = \mathfrak{D}_i$ .*

**Proof :** This proof is based on several intermediate results the proofs of which are postponed to the three following sections. Let  $F$  be a Dirichlet form in  $\mathfrak{D}_i$ . We prove in Theorem 3, that there exists a sequence  $(F_{0,0,\mu_m})$  of continuous objective Dirichlet forms which Mosco-converges to  $F$  for the  $L^2(\Omega)$  topology. Then, we prove in Theorem 4 that, for any  $m$ , there exists a sequence  $(\mu_{m,n})$  of atomic interactions such that  $(F_{0,0,\mu_{m,n}})$  converges to  $F_{0,0,\mu_m}$ . Finally, owing to theorem 6, there exists a sequence  $(F_{\alpha_{m,n,p},0,0})$  in  $\mathfrak{D}_d$  which converges to  $F_{0,0,\mu_{m,n}}$ . Therefore, owing to Remark 5, there exists a diagonal sequence in  $\mathfrak{D}_d$  which converges to  $F$ .  $\square$

**Remark 8** *Owing to remarks 9, 10, 12, Theorem 1 remains valid when replacing the Mosco-convergence for the  $L^2(\Omega)$  topology by the  $\tau$ -convergence.*

As a consequence of the last remark, we prove in section 7 this second result :

**Theorem 2** *The Mosco-closure in the  $L^2(\Omega)$ -topology of the set  $\mathfrak{D}_0$  of diffusion functionals submitted to a Dirichlet boundary condition coincides with the set of all Dirichlet forms :  $\overline{\mathfrak{D}_0} = \mathfrak{D}$ .*

### 3 Moreau-Yosida approximation

Let  $F \in \mathfrak{D}$  be a Dirichlet form and  $\lambda$  a positive real number. The *Moreau-Yosida approximation* of index  $\lambda$  of  $F$  [11] is the functional defined on  $L^2(\Omega)$  by

$$Y_\lambda(F)(u) = \inf_{v \in L^2(\Omega)} \{F(v) + \lambda \|u - v\|_{L^2(\Omega)}^2\}. \quad (3.1)$$

Clearly  $Y_\lambda(F)$  is a Dirichlet form and is locally Lipschitz and then continuous on  $L^2(\Omega)$  :  $Y_\lambda(F) \in \mathfrak{D}_c$ . On the other hand, if  $F \in \mathfrak{D}_i$ , so is its Moreau-Yosida approximation  $Y_\lambda(F)$ .

**Theorem 3** *The sets of continuous Dirichlet forms  $\mathfrak{D}_c$  or objective continuous Dirichlet forms  $\mathfrak{D}_{ci}$  are respectively dense in  $\mathfrak{D}$  and  $\mathfrak{D}_i$  :  $\overline{\mathfrak{D}_c} = \mathfrak{D}$  and  $\overline{\mathfrak{D}_{ci}} = \mathfrak{D}_i$ .*

**Proof :** Consider the sequence  $(Y_n(F))$ . It belongs to  $\mathfrak{D}_c$  (to  $\mathfrak{D}_{ci}$  if  $F \in \mathfrak{D}_i$ ) and

$$\lim_n Y_n(F)(u) = F(u) \quad (3.2)$$

(see, e.g., [11], Remark 9.11). This proves the upper-bound inequality. Indeed, it is enough to choose as an approximating sequence, the constant one  $u_n := u$ .

Now, consider a sequence  $(u_n)$  which converges weakly in  $L^2(\Omega)$  to some  $u$ . For any  $n \geq n_0$ ,

$$Y_n(F)(u_n) \geq Y_{n_0}(F)(u_n). \quad (3.3)$$

Moreover the functional  $Y_{n_0}$  being convex and continuous for the strong topology of  $L^2(\Omega)$ , is lower semicontinuous for the weak topology. Hence , for any  $n_0 \in \mathbb{N}$ ,

$$\liminf_n Y_n(F)(u_n) \geq \liminf_n Y_{n_0}(F)(u_n) \geq Y_{n_0}(F)(u) .$$

Passing to the limit when  $n_0$  tends to infinity in the previous inequality, leads to the lower-bound inequality :  $\liminf_n Y_n(F)(u_n) \geq F(u)$ .  $\square$

The choice we made for the approximating sequence in the upper-bound inequality clearly implies :

**Remark 9** *Theorem 3 remains valid when replacing the Mosco-convergence for the  $L^2(\Omega)$  topology by the  $\tau$ -convergence.*

## 4 Discretization of a non-local interaction

Here we use the concept of atomic interaction we defined in (2.11). We prove that any continuous objective Dirichlet form can be approximated by such interactions :

**Theorem 4** *Let  $F \in \mathfrak{D}_{ci}$ . There exists a sequence  $(\mu_n)$  in  $\mathcal{A}$  such that  $F_{0,0,\mu_n}$  Mosco-converges to  $F$  in the  $L^2(\Omega)$  topology.*

**Proof :** Let us recall that any  $F \in \mathfrak{D}_{ci}$  can be represented by a measure  $\mu$  on  $\Omega \times \Omega$  :

$$F(u) = \int_{\Omega \times \Omega} (u(x) - u(y))^2 \mu(dx, dy).$$

Let  $n$  denote a sequence of integers tending to infinity of the form  $n = 2^{q_n}$  (with  $q_n \in \mathbb{N}$ ). We divide the domain  $\Omega$  in  $n^N$  elementary cubes

$$\Omega_I^n := \left( \frac{i_1 - 1}{n}, \frac{i_1}{n} \right) \times \left( \frac{i_2 - 1}{n}, \frac{i_2}{n} \right) \times \dots \times \left( \frac{i_N - 1}{n}, \frac{i_N}{n} \right), \quad (4.1)$$

of centers  $c_I^n$  where  $I = (i_1, i_2, \dots, i_N)$  belongs to  $\{1 \dots n\}^N$  (which we identify with  $\{1 \dots n^N\}$ ) and we consider the sequence  $(\mu_n)$  defined as follows

$$\mu_n := \sum_{I=1}^{n^N} \sum_{I'=1}^{n^N} a_{II'}^n \mathbf{1}_{\Omega_I^n}(x) \delta_{x+w_{II'}^n}(dy) \mathbf{1}_{\Omega}(x + w_{II'}^n) dx \quad (4.2)$$

where  $a_{II'}^n := n^N \mu(\Omega_I^n \times \Omega_{I'}^n)$  and  $w_{II'}^n := c_{I'}^n - c_I^n$ . Note that, as every vector  $w_{II'}^n$  has dyadic components,  $\mu_n$  belongs to  $\mathcal{A}$ .

Now let us prove that the sequence  $(F_{0,0,\mu_n})$  Mosco-converges to  $F_{0,0,\mu}$ , as  $n$  tends to infinity.

Let  $(u_n)$  be a sequence with bounded energy ( $F_{0,0,\mu_n}(u_n) < M < +\infty$ ) and converging to some  $u$  weakly in  $L^2(\Omega)$ . For any  $n$ , we define the piecewise constant function  $\bar{u}_n$  by

$$\bar{u}_n(x) := \sum_{I=1}^{n^N} \left( \int_{\Omega_I^n} u_n \right) \mathbf{1}_{\Omega_I^n}(x). \quad (4.3)$$

Note that  $(\bar{u}_n)$  converges also to  $u$  weakly in  $L^2(\Omega)$ . The definitions of  $\mu_n$  and  $\bar{u}_n$  imply

$$\begin{aligned}
F_{0,0,\mu_n}(u_n) &= \sum_{I=1}^{n^N} \sum_{I'=1}^{n^N} a_{II'}^n |\Omega_I^n| \int_{\Omega_I^n} [(u_n(x) - u_n(x + w_{II'}^n))]^2 dx \\
&\geq \sum_{I=1}^{n^N} \sum_{I'=1}^{n^N} a_{II'}^n |\Omega_I^n| \left( \int_{\Omega_I^n} u_n(x) dx - \int_{\Omega_I^n} u_n(x + w_{II'}^n) dx \right)^2 \\
&\geq \sum_{I=1}^{n^N} \sum_{I'=1}^{n^N} a_{II'}^n \int_{\Omega_I^n} [(\bar{u}_n(x) - \bar{u}_n(x + w_{II'}^n))]^2 dx \\
&\geq F_{0,0,\mu_n}(\bar{u}_n).
\end{aligned} \tag{4.4}$$

As  $F_{0,0,\mu}$  is continuous for the strong topology of  $L^2(\Omega)$ , the measure  $\mu^{sym}(dx, \Omega)$  does not charge the sets  $\partial\Omega_I^n$ . Hence  $F_{0,0,\mu_n}(\bar{u}_n) = F_{0,0,\mu}(\bar{u}_n)$ . As  $(\bar{u}_n)$  converges to  $u$  weakly and as  $F_{0,0,\mu}$  is lower semicontinuous for the weak topology, we get the lower-bound inequality

$$\liminf_{n \rightarrow \infty} F_{0,0,\mu_n}(u_n) \geq \liminf_{n \rightarrow \infty} F_{0,0,\mu}(\bar{u}_n) \geq F_{0,0,\mu}(u).$$

Now let  $u \in L^2(\Omega)$  such that  $F_{0,0,\mu}(u) \leq M < +\infty$ . By a density argument, we can assume that  $u \in C_0^1(\Omega)$ . We associate to the constant sequence  $u_n := u$  the sequence  $(\bar{u}_n)$  defined by (4.3). As  $(\bar{u}_n)$  converges uniformly to  $u$ , we have

$$F_{0,0,\mu_n}(u_n) = F_{0,0,\mu_n}(\bar{u}_n) + O\left(\frac{1}{n}\right) = F_{0,0,\mu}(\bar{u}_n) + O\left(\frac{1}{n}\right). \tag{4.5}$$

Hence  $\limsup F_{0,0,\mu_n}(u_n) \leq \limsup F_{0,0,\mu}(\bar{u}_n) = F_{0,0,\mu}(u)$  and the upper-bound inequality is proved.  $\square$

The way we chose the approximating sequence  $(u_n)$  clearly shows that :

**Remark 10** *Theorem 4 remains valid when replacing the Mosco-convergence for the  $L^2(\Omega)$  topology by the  $\tau$ -convergence.*

## 5 A crucial step : a homogenization result

Here we prove that any elementary interaction is the Mosco-limit of a sequence of diffusion functionals in  $\mathfrak{D}_d$ . We explicitly construct a diffusive material containing very thin and very high conductivity fibers which lead to the desired effective properties.

**Theorem 5** *Let  $\alpha \in L_{++}^\infty(\Omega)$  or  $\alpha \equiv 0$ , and let  $\mu(dx, dy) := \delta_{x+w}(dy) \mathbf{1}_\Omega(x+w) f(x) dx$  in  $\mathcal{E}$ . Then, there exists a sequence  $(\alpha_n)$  in  $L_{++}^\infty(\Omega)$  such that  $(F_{\alpha_n,0,0})$  Mosco-converges to  $F_{\alpha,0,\mu}$  in the  $L^2(\Omega)$ -topology.*

We describe the heterogeneous material in section 5.1 while we prove in sections 5.2 and 5.3 the Mosco-convergence of  $(F_{\alpha_n,0,0})$  to  $F_{\alpha,0,\mu}$ .

## 5.1 Description of the heterogeneous material

The non-local interaction  $\mu$  is simulated by highly conductive cylinders of axis  $w$ . Let us describe these fibers precisely .

As  $w \in \Omega_2$ , there exists  $q \in \mathbb{N}$  such that  $2^q w \in \mathbb{N}^N$ . We use the notations introduced in section 4. Here  $n$  denotes a sequence of integers tending to infinity (of the form  $n = 2^{q_n}$  with  $q_n > q$ ) and  $(r_n)$  is a sequence of reals tending to zero in such a way that

$$\lim_{n \rightarrow \infty} n^{-N} |\ln r_n| = +\infty. \quad (5.1)$$

We denote  $\mathcal{I}^n$  the set of indices  $I \in \{1, \dots, n^N\}$  such that  $\Omega_I^n + w \subset \Omega$ . Note that, for such indices, due to our assumptions on  $w$  and  $n$ ,  $\Omega_I^n + w$  is again an elementary cube. Let us define the radii of our high conductivity fibers by setting, for any  $I \in \mathcal{I}^n$ ,

$$r_I^n := r_n \left( \frac{1}{n} + \frac{\|w\|}{|\omega_{N-1}|} \int_{\Omega_I^n} f(x) dx \right)^{\frac{1}{N-1}}, \quad (5.2)$$

and introduce

$$R := \left( 4 + \frac{\|w\|}{|\omega_{N-1}|} \|f\|_{L^\infty(\Omega)} \right)^{\frac{1}{N-1}}. \quad (5.3)$$

Thus  $Rr_n$  bounds all radii  $r_I^n$ .

In order to define the end points of our high conductivity fibers, we introduce a family of points  $x_I^n$  in the following way : let  $\Delta_I^n$  be the straight line passing through the point  $x_I^n$  and parallel to  $w$ , and  $p_I^n(x)$  the orthogonal projection of  $x$  on  $\Delta_I^n$ , we assume that the family of points  $x_I^n$  satisfies the three following assumptions

$$\|x_I^n - c_I^n\| < (4n)^{-1}, \quad (5.4)$$

$$\text{if } \Omega_I^n + mw = \Omega_{I'}^n \text{ for some } m \in \mathbb{Z}, \quad \text{then } x_I^n + mw = x_{I'}^n, \quad (5.5)$$

$$d(\Delta_I^n, x_{I'}^n) > 2Rn^{-\frac{N-1}{N-2}}, \quad \text{otherwise} . \quad (5.6)$$

These assumptions avoid any collision between the fibers. In order to prove the existence of such a family, we prove, using an induction argument, that, for any  $p \in \{1 \dots n^N\}$  there exists a family  $\{x_1^n, x_2^n, \dots, x_p^n\}$  which satisfies property  $\mathcal{P}_p$  :

$$\mathcal{P}_p \left\{ \begin{array}{l} \forall I \leq p, \quad \|x_I^n - c_I^n\| < (4n)^{-1}, \\ \forall I, I' \leq p, \quad \text{if } \Omega_I^n + mw = \Omega_{I'}^n \text{ for some } m \in \mathbb{Z}, \quad \text{then } x_I^n + mw = x_{I'}^n, \\ \quad \quad \quad d(\Delta_I^n, x_{I'}^n) > 2Rn^{-\frac{N-1}{N-2}}, \quad \text{otherwise.} \end{array} \right.$$

When  $p = 1$ , it is enough to choose  $x_1^n = c_1^n$ . Now, assume that there exist  $p - 1$  points  $\{x_1^n, x_2^n, \dots, x_{p-1}^n\}$  which satisfy property  $\mathcal{P}_{p-1}$ . If  $\Omega_I^n + mw = \Omega_p^n$  for some  $I < p$  and some  $m \in \mathbb{Z}$ , it is enough to choose  $x_p^n = x_I^n + mw$ . Otherwise, for each  $I \leq p - 1$ , let us consider the cylinder of axis  $\Delta_I^n$  and radius  $2Rn^{-\frac{N-1}{N-2}}$ . It is easy to check that the number of such cylinders which cut the ball  $B(c_p^n, (4n)^{-1})$  is less than  $C_N n$  and that the

volume of each intersection is smaller than  $C'_N n^{-1} n^{-\frac{(N-1)^2}{N-2}}$  for some constants  $C_N$  and  $C'_N$  depending only on the dimension  $N$  of the ambient space. Thus, the total volume of these intersections is smaller than  $C_N C'_N n^{-\frac{(N-1)^2}{N-2}}$  and then smaller than the volume of the ball  $B(c_p^n, (4n)^{-1})$ . Therefore we can choose the point  $x_p^n$  in this ball and outside of all the cylinders. Clearly the family  $\{x_1^n, x_2^n, \dots, x_p^n\}$  fulfills properties  $\mathcal{P}_p$ .  $\square$

The high conductivity fibers are the cylinders  $\mathcal{C}_I^n$ , of radius  $r_I^n$ , of axis  $\Delta_I^n$  and length  $\|w\|$  :

$$\mathcal{C}_I^n := \{ x \in \Omega \mid p_I^n(x) \in [x_I^n, x_I^n + w], \|x - p_I^n(x)\| \leq r_I^n \}. \quad (5.7)$$

As the radii of the cylinders are very small, they are weakly connected with the matrix. In order to improve this connection (at the extremities only), we add high conductivity balls

$$\mathcal{B}_I^n := B \left( x_I^n, n^{-\frac{N-1}{N-2}} \right). \quad (5.8)$$

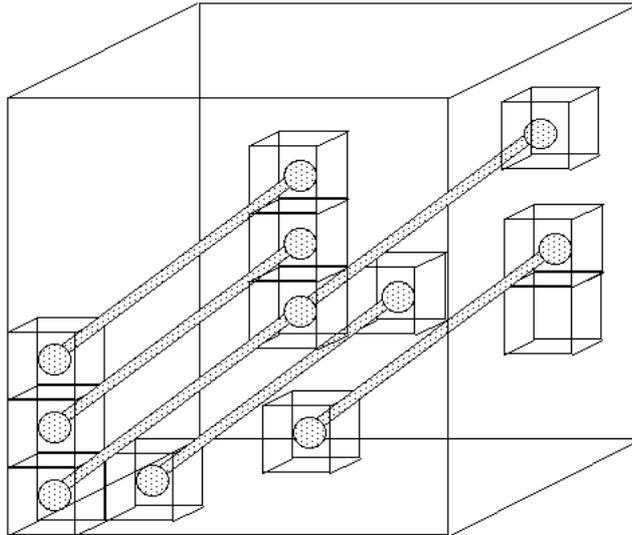
Then we define the high conductivity part  $\Omega^n$  of the material (see figure 1) by :

$$\Omega^n := \left( \bigcup_{I \in \mathcal{I}^n} \mathcal{C}_I^n \right) \cup \left( \bigcup_I \mathcal{B}_I^n \right). \quad (5.9)$$

The conductivity coefficient in  $\Omega^n$  is assumed to be constant equal to  $r_n^{-(N-1)} n^{-N}$ . Hence, the conductivity coefficient of the composite material in consideration is defined by :

$$\alpha_n(x) := \begin{cases} \alpha(x) + n^{-1/2} & \text{if } x \in \Omega \setminus \Omega^n, \\ r_n^{-(N-1)} n^{-N} & \text{if } x \in \Omega^n. \end{cases} \quad (5.10)$$

Note that the addition of the term  $n^{-1/2}$  to  $\alpha$  in the matrix is needed only for  $\alpha \equiv 0$  and ensures that  $\alpha_n$  belongs to  $L_{++}^\infty(\Omega)$ .



**Figure 1 :** Geometry of the composite material.

## 5.2 Lower-bound inequality

Let  $(u_n)$  be a sequence with bounded energy ( $F_{\alpha_n,0,0}(u_n) < M$ ) converging to  $u$  weakly in  $L^2(\Omega)$ .

Let us first estimate the energy of  $u_n$  in the matrix  $\Omega \setminus \Omega^n$ . As  $\int_{\Omega^n} r_n^{1-N} n^{-N} |\nabla u_n|^2 dx < M$ , the quantity  $\int_{\Omega^n} \alpha(x) |\nabla u_n|^2 dx$  tends to zero. Hence

$$\liminf_{n \rightarrow \infty} \int_{\Omega \setminus \Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \alpha(x) |\nabla u_n|^2 dx \geq \liminf_{n \rightarrow \infty} F_{\alpha,0,0}(u_n). \quad (5.11)$$

As  $F_{\alpha,0,0}$  is lower semicontinuous for the weak topology of  $L^2(\Omega)$ , we get

$$\liminf_{n \rightarrow \infty} \int_{\Omega \setminus \Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \geq F_{\alpha,0,0}(u). \quad (5.12)$$

Now let us estimate the energy of  $u_n$  in  $\Omega^n$ . For every  $I \in \mathcal{I}^n$ , we define  $\mathcal{D}_I^n := \{x \in \mathcal{C}_I^n, \|p_I^n(x) - x_I^n\| \leq n^{-\frac{N-1}{N-2}}/2\}$  and  $\tilde{\mathcal{D}}_I^n := \{x \in \mathcal{C}_I^n, \|p_I^n(x) - x_I^n\| \leq n^{-\frac{N-1}{N-2}}/2\}$  where  $\tilde{I}$  denotes the index of point  $x_I^n + w$  ( $\Omega_{\tilde{I}}^n = \Omega_I^n + w$ ). Thus  $\mathcal{D}_I^n$  and  $\tilde{\mathcal{D}}_I^n$  are two extremity parts of the cylinder  $\mathcal{C}_I^n$ . In order to estimate the energy of  $u_n$  in  $\mathcal{C}_I^n$ , let us temporarily use the ‘‘cylindrical’’ coordinates  $(x', x_N)$  with center  $x_I^n$  and axis  $w$ . Recalling that  $\omega_{N-1}$  denotes the unit ball of  $\mathbb{R}^{N-1}$ , the cylinder  $\mathcal{C}_I^n$  reads  $\{(x', x_N) \in r_I^n \omega_{N-1} \times [0, \|w\|]\}$ . For every  $0 < y < z < \|w\|$ , we have

$$\begin{aligned} \int_{\mathcal{C}_I^n} |\nabla u_n|^2 dx &= \int_{r_I^n \omega_{N-1}} \left( \int_0^{\|w\|} |\nabla u_n|^2 dx_N \right) dx' \\ &\geq \int_{r_I^n \omega_{N-1}} \left( \int_0^{\|w\|} \left( \frac{\partial u_n}{\partial x_N} \right)^2 dx_N \right) dx' \geq \int_{r_I^n \omega_{N-1}} \left( \int_y^z \left( \frac{\partial u_n}{\partial x_N} \right)^2 dx_N \right) dx' \\ &\geq \frac{1}{z-y} \int_{r_I^n \omega_{N-1}} \left( \int_y^z \frac{\partial u_n}{\partial x_N} dx_N \right)^2 dx' \geq \frac{1}{\|w\|} \int_{r_I^n \omega_{N-1}} (u_n(x', z) - u_n(x', y))^2 dx'. \end{aligned} \quad (5.13)$$

Taking the mean value of this last term for  $y \in [0, \frac{1}{2}n^{-\frac{N-1}{N-2}}]$  and  $z \in [\|w\| - \frac{1}{2}n^{-\frac{N-1}{N-2}}, \|w\|]$  and using Jensen inequality, we get

$$\begin{aligned} \int_{\mathcal{C}_I^n} |\nabla u_n|^2 dx &\geq \frac{1}{\|w\|} \int_{r_I^n \omega_{N-1}} \left( \int_0^{\frac{1}{2}n^{-\frac{N-1}{N-2}}} \left( \int_{\|w\| - \frac{1}{2}n^{-\frac{N-1}{N-2}}}^{\|w\|} (u_n(x', z) - u_n(x', y))^2 dz \right) dy \right) dx' \\ &\geq \frac{|\omega_{N-1}| (r_I^n)^{N-1}}{\|w\|} \left( \int_{\mathcal{D}_I^n} u_n - \int_{\tilde{\mathcal{D}}_I^n} u_n \right)^2. \end{aligned}$$

Hence

$$\int_{\Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \geq \frac{|\omega_{N-1}|}{\|w\|} \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \frac{r_I^n}{r_n} \right)^{N-1} \left( \int_{\mathcal{D}_I^n} u_n - \int_{\tilde{\mathcal{D}}_I^n} u_n \right)^2. \quad (5.14)$$

Noticing that  $\mathcal{D}_I^n$  are contained in balls  $\mathcal{B}_I^n$ , Poincaré-Wirtinger inequality applied to the ball gives

$$\int_{\mathcal{D}_I^n} (u_n - \fint_{\mathcal{B}_I^n} u_n)^2 dx \leq \int_{\mathcal{B}_I^n} (u_n - \fint_{\mathcal{B}_I^n} u_n)^2 dx \leq \frac{C}{n^{\frac{2(N-1)}{N-2}}} \int_{\mathcal{B}_I^n} |\nabla u_n|^2 dx.$$

As the volume of  $\mathcal{D}_I^n$  is larger than  $|\omega_{N-1}| \frac{1}{n} (r_n)^{N-1} \frac{1}{2} n^{-\frac{N-1}{N-2}}$ , using Jensen inequality and summing over  $I$ , we obtain

$$\left( \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \fint_{\mathcal{D}_I^n} u_n - \fint_{\mathcal{B}_I^n} u_n \right)^2 \right)^{1/2} \leq \left( \frac{2Cn^{-\frac{1}{N-2}}}{|\omega_{N-1}|} \sum_{I \in \mathcal{I}^n} \int_{\mathcal{B}_I^n} \alpha_n(x) |\nabla u_n|^2 dx \right)^{1/2} \leq \left( \frac{2CM}{|\omega_{N-1}|} \right)^{1/2} n^{-\frac{1}{2(N-2)}}. \quad (5.15)$$

In the same way, as  $\tilde{\mathcal{D}}_I^n \subset \mathcal{B}_I^n \subset \Omega_I^n + w$ , we have

$$\left( \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \fint_{\tilde{\mathcal{D}}_I^n} u_n - \fint_{\mathcal{B}_I^n} u_n \right)^2 \right)^{1/2} \leq \left( \frac{2Cn^{-\frac{1}{N-2}}}{|\omega_{N-1}|} \sum_{I \in \mathcal{I}^n} \int_{\mathcal{B}_I^n} \alpha_n(x) |\nabla u_n|^2 dx \right)^{1/2} \leq \left( \frac{2CM}{|\omega_{N-1}|} \right)^{1/2} n^{-\frac{1}{2(N-2)}}. \quad (5.16)$$

Let us denote by  $\mathcal{F}_I^n$  the set  $\mathcal{F}_I^n := \{x \in \Omega; \frac{1}{8n} < \|x - x_I^n\| < \frac{1}{4n}\}$ , and let us use the adapted coordinates  $\rho := \|x - x_I^n\|^{2-N}$ ,  $y := \rho^{\frac{1}{N-2}}(x - x_I^n)$  ( $y$  is an element of the unit sphere  $S$  of  $\mathbb{R}^N$ ). Let us introduce the measure  $\eta$  on  $\mathbb{R}^+$  by :

$$\eta(d\rho) := \frac{1}{N-2} \rho^{-2\frac{N-1}{N-2}} d\rho \quad (5.17)$$

and the mean value with respect to this measure by setting for any Borel set  $A \subset \mathbb{R}$

$$\fint_A f(s) \eta(ds) := \left( \int_A \eta(ds) \right)^{-1} \left( \int_A f(s) \eta(ds) \right). \quad (5.18)$$

Noting that  $\eta(d\rho)dy$  is the volume measure for the considered coordinates, we have

$$\begin{aligned} \left( \fint_{\mathcal{B}_I^n} u_n - \fint_{\mathcal{F}_I^n} u_n \right)^2 &= \left( \fint_{y \in S} \fint_{n^{N-1}}^\infty u_n(x_I^n + s^{\frac{1}{2-N}} y) \eta(ds) dy - \fint_{y \in S} \fint_{(4n)^{N-2}}^{(8n)^{N-2}} u_n(x_I^n + t^{\frac{1}{2-N}} y) \eta(dt) dy \right)^2 \\ &\leq \fint_{y \in S} \fint_{n^{N-1}}^\infty \fint_{(4n)^{N-2}}^{(8n)^{N-2}} \left( u_n(x_I^n + s^{\frac{1}{2-N}} y) - u_n(x_I^n + t^{\frac{1}{2-N}} y) \right)^2 \eta(dt) \eta(ds) dy \\ &\leq \fint_{y \in S} \fint_{n^{N-1}}^\infty \fint_{(4n)^{N-2}}^{(8n)^{N-2}} \left( \int_t^s |\nabla u_n(x_I^n + r^{\frac{1}{2-N}} y)| \frac{1}{2-N} r^{\frac{N-1}{2-N}} dr \right)^2 \eta(dt) \eta(ds) dy \\ &\leq \fint_{y \in S} \fint_{n^{N-1}}^\infty \fint_{(4n)^{N-2}}^{(8n)^{N-2}} \left( \int_t^s dr \int_t^s |\nabla u_n(x_I^n + r^{\frac{1}{2-N}} y)|^2 \eta(dr) \right) \eta(dt) \eta(ds) dy \\ &\leq \frac{1}{N|\omega_N|} \int_{\Omega_I^n} |\nabla u_n(x)|^2 dx \fint_{n^{N-1}}^\infty \fint_{(4n)^{N-2}}^{(8n)^{N-2}} (s-t) \eta(dt) \eta(ds) \\ &\leq \frac{n^{N-1}}{2|\omega_N|} \int_{\Omega_I^n} |\nabla u_n(x)|^2 dx. \end{aligned}$$

Since  $\alpha_n > n^{-1/2}$ ,  $\int_{\Omega} |\nabla u_n(x)|^2 dx \leq M\sqrt{n}$ . By summing over  $I$  the last inequalities we obtain

$$\left( \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \int_{B_I^n} u_n - \int_{\mathcal{F}_I^n} u_n \right)^2 \right)^{1/2} \leq \left( \frac{M}{2|\omega_N|\sqrt{n}} \right)^{1/2}. \quad (5.19)$$

On the other hand, Poincaré-Wirtinger inequality applied to  $\Omega_I^n$  gives

$$\int_{\mathcal{F}_I^n} (u_n - \int_{\Omega_I^n} u_n)^2 dx \leq \int_{\Omega_I^n} (u_n - \int_{\Omega_I^n} u_n)^2 dx \leq \frac{C'''}{n^2} \int_{\Omega_I^n} |\nabla u_n|^2 dx. \quad (5.20)$$

As  $\alpha_n > n^{-1/2}$ , the sum over  $I$  of the last inequalities leads to

$$\left( \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \int_{\mathcal{F}_I^n} u_n - \int_{\Omega_I^n} u_n \right)^2 \right)^{1/2} \leq \left( \frac{C''' M}{n^{3/2}} \right)^{1/2}. \quad (5.21)$$

Collecting (5.15), (5.16), (5.19) and (5.21), the estimate (5.14) becomes

$$\left( \int_{\Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \right)^{1/2} \geq \left( \frac{|\omega_{N-1}|}{\|\mathbf{w}\|} \frac{1}{n^N} \sum_{I \in \mathcal{I}^n} \left( \frac{r_I^n}{r_n} \right)^{N-1} \left( \int_{\Omega_I^n} u_n - \int_{\mathcal{F}_I^n} u_n \right)^2 \right)^{1/2} - O(n^{-\frac{1}{4}}) - O(n^{\frac{-1}{2(N-2)}}). \quad (5.22)$$

Recalling definitions (5.2) of  $r_I^n$ , and (4.3) of  $\bar{u}_n$ , the last inequality reads

$$\begin{aligned} \left( \int_{\Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \right)^{1/2} &\geq \left( \int_{\Omega} (\bar{u}_n(x) - \bar{u}_n(x + \mathbf{w}))^2 f(x) dx \right)^{1/2} - O(n^{-\frac{1}{4}}) - O(n^{\frac{-1}{2(N-2)}}) \\ &\geq (F_{0,0,\mu}(\bar{u}_n))^{1/2} - O(n^{-\frac{1}{4}}) - O(n^{\frac{-1}{2(N-2)}}). \end{aligned} \quad (5.23)$$

As  $f$  belongs to  $L^\infty(\Omega)$ , and as the convergence of  $(\bar{u}_n)$  to  $u$  is obtained by a new application of Poincaré-Wirtinger inequality, we can pass to the limit in (5.23) and get

$$\liminf_{n \rightarrow \infty} \int_{\Omega^n} \alpha_n(x) |\nabla u_n|^2 dx \geq \int_{\Omega} (u(x) - u(x + \mathbf{w}))^2 f(x) dx. \quad (5.24)$$

Inequalities (5.12) and (5.24) imply the lower-bound inequality :

$$\liminf_{n \rightarrow \infty} F_{\alpha_n, 0, 0}(u^n) \geq F_{\alpha, 0, \mu}(u). \quad (5.25)$$

□

### 5.3 Upper-bound inequality

Let  $u \in L^2(\Omega)$  such that  $F_{\alpha, 0, \mu}(u) < \infty$ . As  $C^1(\bar{\Omega})$  is dense in the domain of  $F_{\alpha, 0, \mu}$  for the norm  $\sqrt{\|u\|_{L^2(\Omega)}^2 + F_{\alpha, 0, \mu}(u)}$ , it is enough to prove the upper-bound inequality when  $u \in C^1(\bar{\Omega})$ .

We construct the approximating sequence  $(u_n)$  in two steps. First we define the sequence  $(\tilde{u}_n)$  by

$$\tilde{u}_n(x) := \begin{cases} u(x_I^n), & \text{if } \|x - x_I^n\| \leq Rn^{-\frac{N-1}{N-2}}, \\ u(x), & \text{if } \|x - x_I^n\| \geq 2Rn^{-\frac{N-1}{N-2}}, \\ u(x) + \frac{1}{2^{N-2} - 1} \left( \frac{(2R)^{N-2}}{n^{N-1} \|x - x_I^n\|^{N-2}} - 1 \right) (u(x_I^n) - u(x)), & \text{otherwise,} \end{cases} \quad (5.26)$$

where  $R$  is the quantity defined by (5.3). It is easy to verify that  $\tilde{u}_n$  is continuous on  $\Omega$ , constant on the balls  $B(x_I^n, Rn^{-\frac{N-1}{N-2}})$  (which contain the balls  $\mathcal{B}_I^n$ ), coincides with  $u$  on  $\mathcal{B}$  and satisfies  $\|\tilde{u}_n\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ . Moreover, since

$$|u(x) - u(x_I^n)| \leq \sqrt{N}n^{-1} \|\nabla u\|_{L^\infty(\Omega)} \quad \forall x \in \Omega_I^n, \quad (5.27)$$

the sequence  $(\tilde{u}_n)$  converges uniformly to  $u$ . Let us denote  $\mathcal{G}_I^n := B(x_I^n, 2Rn^{-\frac{N-1}{N-2}}) \setminus B(x_I^n, Rn^{-\frac{N-1}{N-2}})$  the transition layers. A straightforward computation leads to the following estimation (in which  $C_N$  is a constant depending only on  $N$ ):

$$\sum_{I=1}^{n^N} \int_{\mathcal{G}_I^n} |\nabla \tilde{u}_n(x)|^2 dx \leq C_N R^N n^{\frac{-N}{N-2}} \|\nabla u\|_{L^\infty(\Omega)}^2. \quad (5.28)$$

As, outside the transition layers  $\mathcal{G}_I^n$ , the function  $\tilde{u}_n$  is either constant, or equal to  $u$ , we obtain, for any sequence of domains  $\mathcal{G}^n$  the volume of which tends to zero,

$$\lim_{n \rightarrow \infty} \int_{\mathcal{G}^n} |\nabla \tilde{u}_n(x)|^2 dx = 0. \quad (5.29)$$

The domain where the function  $\tilde{u}_n$  does not coincide with  $u$  has a volume which tends to zero. Then the last remark implies that the sequence  $(\tilde{u}_n)$  tends to  $u$  for the  $H^1(\Omega)$  norm. Hence

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \left( \alpha(x) + \frac{1}{\sqrt{n}} \right) |\nabla \tilde{u}_n|^2 dx = \int_{\Omega} \alpha(x) |\nabla u|^2 dx. \quad (5.30)$$

Now, let us define the non-decreasing continuous interpolation functions  $f_n$  and  $g_n$ , from  $\mathbb{R}$  to  $[0, 1]$ , by

$$f_n(s) = \frac{\log(s)}{|\log(n^{\frac{N-1}{N-2}} r_n)|} \quad \forall s \in (1, r_n^{-1} n^{-\frac{N-1}{N-2}}) \quad \text{and} \quad g_n(s) = \frac{s - n^{-\frac{N-1}{N-2}}}{\|w\| - 2n^{-\frac{N-1}{N-2}}} \quad \forall s \in (n^{-\frac{N-1}{N-2}}, \|w\| - n^{-\frac{N-1}{N-2}}).$$

Note that  $f_n(s) = 0$  for  $s \leq 1$ ,  $g_n(s) = 0$  for  $s \leq n^{-\frac{N-1}{N-2}}$ ,  $f_n(s) = 1$  for  $s \geq r_n^{-1} n^{-\frac{N-1}{N-2}}$  and  $g_n(s) = 1$  for  $s \geq \|w\| - n^{-\frac{N-1}{N-2}}$ . In each cylinder ( $I \in \mathcal{I}^n$ )

$$\mathcal{A}_I^n := \left\{ x \in \Omega, p_I^n(x) \in ]x_I^n, x_I^n + w[, \|x - p_I^n(x)\| < Rn^{-\frac{N-1}{N-2}} \right\}, \quad (5.31)$$

let us use the adapted ‘‘cylindrical’’ coordinates  $(r, z)$  with center  $x_I^n$  and axis  $\Delta_I^n$  ( $r := \|x - p_I^n(x)\|$ ,  $z := \|p_I^n(x) - x_I^n\|$ ). Let us denote  $\mathcal{A}^n := \bigcup_{I \in \mathcal{I}^n} \mathcal{A}_I^n$ . We define the approximating sequence  $(u_n)$  by setting for any  $I$  in  $\mathcal{I}^n$  and any  $x$  in  $\mathcal{A}_I^n$

$$u_n(x) = f_n\left(\frac{r(x)}{r_I^n}\right) \tilde{u}^n(x) + \left(1 - f_n\left(\frac{r(x)}{r_I^n}\right)\right) \left( (1 - g_n(z(x))) u(x_I^n) + g_n(z(x)) u(x_I^n + w) \right) \quad (5.32)$$

and by setting  $u_n(x) = \tilde{u}_n(x)$  in the remaining part  $\Omega \setminus \mathcal{A}^n$  of  $\Omega$ . Due to assumption (5.6), the sets  $\mathcal{A}_I^n$  are disjointed and  $u_n$  is well defined by (5.32). For an analogous reason, note also that

$$u = 0 \text{ on } \mathcal{B} \implies u_n = 0 \text{ on } \mathcal{B}. \quad (5.33)$$

It is easy to check that the definitions of  $R$  and  $f_n$ , and the way we defined  $\tilde{u}_n$  assure the continuity of  $u_n$  in the whole domain  $\Omega$ .

Now let us estimate the energy of  $u_n$  on the different parts of the domain : the reinforcing set  $\Omega^n$ , the transition zone  $\mathcal{A}^n \setminus \Omega^n$  and the remaining part  $\Omega \setminus \mathcal{A}^n$ . On each ball  $\mathcal{B}_I^n$ , owing to the definition of  $g_n$ , the functions  $u_n$  are constant. Then, a straightforward computation gives, for the reinforcing part energy,

$$\begin{aligned} \int_{\Omega^n} \alpha_n(x) |\nabla u_n(x)|^2 dx &= \sum_{I \in \mathcal{I}^n} \int_{\mathcal{C}_I^n} \alpha_n(x) |\nabla u_n(x)|^2 dx \\ &\leq |\omega_{N-1}| r_n^{1-N} n^{-N} \|w\|^{-1} \sum_{I \in \mathcal{I}^n} \left( (r_I^n)^{N-1} (u(x_I^n) - u(x_I^n + w))^2 \right) (1 + O(n^{-\frac{N-1}{N-2}})). \end{aligned} \quad (5.34)$$

Using (5.2), (5.27) and passing to the limit as  $n$  tends to infinity, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\Omega^n} \alpha_n(x) |\nabla u_n(x)|^2 dx &\leq n^{-N} \sum_{I=1}^{n^N} \int_{\Omega_I^n} (u(x) - u(x+w))^2 f(x) \mathbf{1}_{\Omega}(x+w) dx \\ &\leq \int_{\Omega \times \Omega} (u(x) - u(y))^2 \delta_{x+w}(y) f(x) dx. \end{aligned} \quad (5.35)$$

On the part  $\Omega \setminus \mathcal{A}^n$ , using (5.30), we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\Omega \setminus \mathcal{A}^n} \alpha_n(x) |\nabla u_n(x)|^2 dx &= \limsup_{n \rightarrow \infty} \int_{\Omega \setminus \mathcal{A}^n} \left( \alpha(x) + \frac{1}{\sqrt{n}} \right) |\nabla \tilde{u}_n(x)|^2 dx \\ &\leq \int_{\Omega} \alpha(x) |\nabla u(x)|^2 dx. \end{aligned} \quad (5.36)$$

Finally, let us estimate the energy on the set  $\mathcal{A}^n \setminus \Omega^n$ . Noting that  $|\mathcal{A}^n|$  tends to zero, that  $g'_n$  is uniformly bounded and using (5.29) we obtain the estimate

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{I=1}^{n^N} \int_{\mathcal{A}_I^n} |\nabla u_n(x)|^2 dx &\leq 12 \|u\|_{L^\infty(\Omega)}^2 \limsup_{n \rightarrow \infty} \sum_{I=1}^{n^N} (r_I^n)^{-2} \int_{\mathcal{A}_I^n} \left( f'_n\left(\frac{r(x)}{r_I^n}\right) \right)^2 dx \\ &\leq 12 \|u\|_{L^\infty(\Omega)}^2 \limsup_{n \rightarrow \infty} \sum_{I=1}^{n^N} \frac{\|w\| |\omega_{N-1}|}{(N-1) \left( \log(n^{\frac{N-1}{N-2}} r_n) \right)^2} \int_{r_I^n}^{Rn^{-\frac{N-1}{N-2}}} r^{N-4} dr. \end{aligned} \quad (5.37)$$

Computing the last integral, we get when  $N > 3$ ,

$$\limsup_{n \rightarrow \infty} \int_{\mathcal{A}^n} |\nabla u_n(x)|^2 dx \leq \limsup_{n \rightarrow \infty} \frac{12 \|u\|_{L^\infty(\Omega)}^2 \|w\| R^{N-3} |\omega_{N-1}| n^N}{(N-1)(N-3) \left(\log(n^{\frac{N-1}{N-2}} r_n)\right)^2} n^{-\frac{(N-1)(N-3)}{N-2}} = 0, \quad (5.38)$$

while, in the particular case  $N = 3$ ,

$$\limsup_{n \rightarrow \infty} \int_{\mathcal{A}^n} |\nabla u_n(x)|^2 dx \leq \limsup_{n \rightarrow \infty} \frac{6\pi n^3 \|u\|_{L^\infty(\Omega)}^2 \|w\| \left(\frac{1}{2} \log(n) - \log(r_n)\right)}{(2 \log(n) + \log(r_n))^2} = 0. \quad (5.39)$$

In these estimations, the assumption (5.1) namely  $n^{-N} |\log(r_n)| \rightarrow \infty$  is fundamental. Note that a weaker assumption would be sufficient in the case  $N > 3$ . In any case, we obtain

$$\limsup_{n \rightarrow \infty} \int_{\mathcal{A}^n \setminus \Omega^n} \alpha_n(x) |\nabla u_n(x)|^2 dx \leq \limsup_{n \rightarrow \infty} \left( \|\alpha + \frac{1}{\sqrt{n}}\|_{L^\infty(\Omega)} \int_{\mathcal{A}^n \setminus \Omega^n} |\nabla u_n(x)|^2 dx \right) = 0. \quad (5.40)$$

Collecting the estimates (5.35), (5.36) and (5.40) we obtain

$$\limsup_{n \rightarrow \infty} F_{\alpha_n, 0, 0}(u_n) \leq F_{\alpha, 0, \mu}(u). \quad (5.41)$$

We have noticed that  $(\tilde{u}_n)$  converges to  $u$  for the  $H^1(\Omega)$  norm, that  $u_n$  coincides with  $\tilde{u}_n$  outside the set  $\mathcal{A}^n$  the volume of which tends to zero, and that  $\lim \int_{\mathcal{A}^n} |\nabla u_n|^2 dx = 0$  (cf. (5.38)-(5.39)). Therefore the convergence of the sequence  $(u_n)$  to  $u$  is assured for the strong topology of  $H^1(\Omega)$  and then for the strong topology of  $L^2(\Omega)$ .

The upper-bound inequality is then proved. This, together with the lower-bound inequality (5.25), concludes the proof of Theorem 5.  $\square$

**Remark 11** *Theorem 5 remains valid when replacing the Mosco-convergence for the  $L^2(\Omega)$  topology by the  $\tau$ -convergence.*

**Proof :** We have to prove that, if  $u \in H_B^1(\Omega)$ , the approximating sequence  $(u_n)$  we defined in the last proof converges to  $u$  strongly in  $H_B^1(\Omega)$ . By a density argument we still can restrict our attention to the case of a function  $u \in C^1(\Omega)$  vanishing on  $\mathcal{B}$ . We have already mentioned that the sequence  $(u_n)$  converges to  $u$  for the strong topology of  $H^1(\Omega)$ . Owing to (5.33), it converges also for the strong topology of  $H_B^1(\Omega)$ .  $\square$

## 6 Extension to atomic interactions

Theorem 5 can easily be extended to a finite sum of elementary interactions. We have :

**Theorem 6** *Let  $\mu$  in  $\mathcal{A}$ . Then, there exists a sequence  $(\alpha_n)$  in  $L^{\infty}_{++}(\Omega)$  such that  $(F_{\alpha_n,0,0})$  Mosco-converges to  $F_{0,0,\mu}$  in the  $L^2(\Omega)$  topology.*

**Proof :** Let us prove, by an induction argument with respect to  $p$ , that the sum of  $p$  elementary interactions :  $\mu := \sum_{i=1}^p \delta_{x+w_i}(dy) f_i(x) \mathbf{1}_{\Omega}(x+w_i) dx$  belongs to the closure  $\overline{\mathfrak{D}}_d$  of  $\mathfrak{D}_d$ .

The case  $p = 1$  is stated by Theorem 5.

Now, for all  $i \leq p$ , let us denote  $\mu^i := \delta_{x+w_i}(dy) f_i(x) \mathbf{1}_{\Omega}(x+w_i) dx$  and  $\tilde{\mu} := \sum_{i=1}^{p-1} \mu^i$ . Assume that the property holds with  $p - 1$ . Then, there exists a sequence  $(F_{\beta_n,0,0})$  in  $\mathfrak{D}_d$  which Mosco-converges to  $F_{0,0,\tilde{\mu}}$  as  $n$  tends to infinity. As the functional  $F_{0,0,\mu^p}$  is convex and continuous for the strong topology of  $L^2(\Omega)$ , owing to Remark 3, we know that  $F_{\beta_n,0,\mu^p} = F_{\beta_n,0,0} + F_{0,0,\mu^p}$  Mosco-converges to  $F_{0,0,\tilde{\mu}} + F_{0,0,\mu^p} = F_{0,0,\mu}$ . Theorem 5 assures the existence, for any  $n \in \mathbb{N}$ , of a sequence  $(F_{\beta_n,m,0,0})$  in  $\mathfrak{D}_d$  which Mosco-converges as  $m$  tends to infinity, to  $F_{\beta_n,0,\mu^p}$ . Then, using a diagonalization argument (cf. Remark 5), there exists a sequence in  $\mathfrak{D}_d$  which Mosco-converges to  $F_{0,0,\mu}$ .  $\square$

**Remark 12** *Theorem 6 remains valid when replacing the Mosco-convergence for the  $L^2(\Omega)$  topology by the  $\tau$ -convergence.*

Indeed, owing to Remark 11, every argument involved in the last proof remains valid for the  $H^1_{\mathcal{B}}(\Omega)$  topology.

## 7 Proof of theorem 2

Let  $F$  in  $\mathfrak{D}$ . We construct a sequence  $(F_{\alpha_n,0,0}^0)$  in  $\mathfrak{D}_0$  which Mosco-converges to  $F$ . Owing to Theorem 3 and diagonalization remark 5, it is enough to study the case  $F \in \mathfrak{D}_c$ . Then  $F$  takes the form  $F = F_{0,\nu,\mu}$ . The point is that the killing term  $\nu$  is equivalent to a non-local interaction between  $\Omega$  and the part  $\mathcal{B}$  of the boundary submitted to the Dirichlet condition : indeed, consider the Radon measure  $\tilde{\mu}$  on  $\overline{\Omega} \times \overline{\Omega}$  defined by  $(\mathcal{H}^2_{|\mathcal{B}}$  denoting the 2-D Hausdorff measure on  $\mathcal{B}$ )

$$\tilde{\mu}(dx, dy) := \mu(dx, dy) + \nu(dx) \mathcal{H}^2_{|\mathcal{B}}(dy), \quad (7.1)$$

and, slightly extending Definition (2.7), denote for any  $u \in L^2(\Omega)$  and  $\lambda > 0$

$$F_{\lambda,0,\tilde{\mu}}(u) := \begin{cases} \int_{\Omega} \lambda |\nabla u(x)|^2 dx + \int_{\overline{\Omega} \times \overline{\Omega}} (u(x) - u(y))^2 \tilde{\mu}(dx, dy), & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \quad (7.2)$$

It is easy to check that, for any  $u \in L^2(\Omega)$ ,

$$F_{0,\nu,\mu}(u) + 2\lambda F_{1,0,0}^0(u) = F_{\lambda,0,\tilde{\mu}}(u) + \lambda F_{1,0,0}^0(u) \quad (7.3)$$

As  $\tilde{\mu}$  does not charge polar sets, one can verify that  $F_{\lambda,0,\tilde{\mu}}$  is a Dirichlet form. It belongs to  $\mathfrak{D}_i$  and then, owing to Theorem 1 and Remark 8, it is the  $\tau$ -limit of some sequence  $(F_{\alpha_n,0,0})$  in  $\mathfrak{D}_d$ . Using Remark 4, we get

$$F_{\alpha_n+\lambda,0,0}^0 = F_{\alpha_n,0,0} + \lambda F_{1,0,0}^0 \xrightarrow{L^2-M} F_{\lambda,0,\tilde{\mu}} + \lambda F_{1,0,0}^0 = F_{0,\nu,\mu} + 2\lambda F_{1,0,0}^0. \quad (7.4)$$

Hence, for any  $\lambda > 0$ , the functional  $F_{0,\nu,\mu} + 2\lambda F_{1,0,0}^0$  belongs to  $\overline{\mathfrak{D}_0}$ . Finally, let us notice that the sequence  $(\frac{1}{n} F_{1,0,0}^0)$  Mosco-converges to zero in the  $L^2(\Omega)$  topology, and that  $F_{0,\nu,\mu}$  is convex and continuous for the strong topology of  $L^2(\Omega)$ . Then Remark 2.16 implies that the sequence  $(F_{0,\nu,\mu} + \frac{1}{n} F_{1,0,0}^0)$  Mosco-converges to  $F_{0,\nu,\mu}$ . Proof of Theorem 2 is completed using the diagonalization procedure stated in Remark 5.  $\square$

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