

Non-local interactions in the Homogenization Closure of Thermoelectric Functionals

M. CAMAR-EDDINE * AND G.W MILTON

Department of Mathematics, University of Utah
155 S 1400 E, Salt Lake City, UT 84112-0090
USA

E-mail: camar@math.utah.edu, milton@math.utah.edu

Abstract

It is a very well known fact that the effective properties of a heterogeneous electrical medium may contain a non-local term. The same phenomenon can occur within a heterogeneous thermally conducting medium. We are interested in the set of all non-local interactions which may arise from the homogenization of a thermoelectric medium where there are couplings between the temperature and the electric field. We consider a bounded open subset $\Omega \subset \mathbb{R}^3$ and we show that any non-local energy of the kind

$$F(u, v) = \int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \boldsymbol{\mu}(dx, dy) \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix},$$

belongs to the closure of the set of thermoelectric functionals, provided that the positive definite symmetric matrix-valued measure $\boldsymbol{\mu}(dx, dy)$ makes this energy functional continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$.

Key words: Homogenization, Gamma-convergence, Mosco-convergence, Composite Materials, Thermoelectricity, Non-local Phenomena.

1 Introduction

In mathematics as well as in physics, it is of great interest to have a complete description of the set of all limits of sequences of the type

$$J_n(u) := \int_{\Omega} w_n(x, \nabla u(x)) dx, \quad (1.1)$$

where $w_n(x, \cdot)$ is a non-negative quadratic form on the space \mathbb{R}^3 (resp. on the space of 3×3 matrices) in the electric or thermal setting (resp. in the linear elasticity setting). In order to place this paper in context, let us review a few of the previous works on asymptotic studies of problems of the type (1.1).

In the case of isotropic dielectric materials the energy density w_n takes the particular form

$$w_n(x, \nabla u(x)) = a_n(x) |\nabla u(x)|^2, \quad (1.2)$$

*The corresponding author.

where the coefficient $a_n \in L^\infty(\Omega, \mathbb{R}^+)$ and $a_n^{-1} \in L^\infty(\Omega, \mathbb{R}^+)$. It is very well known [33] that if the coefficients a_n and their inverses a_n^{-1} are uniformly bounded then the limit functional J^{hom} still takes the same form

$$J^{\text{hom}}(u) = \int_{\Omega} \nabla u(x) \cdot \mathbf{A}^{\text{hom}}(x) \nabla u(x) dx. \quad (1.3)$$

Here the homogenized matrix $\mathbf{A}^{\text{hom}}(x)$ may be anisotropic, *i.e.*, it is not necessarily proportional to the identity tensor (higher order gradient terms do make appearance, but only in the corrector terms to the homogenized functional [3, 30]). The homogenization result (1.3) still holds true under weaker assumptions. Indeed, BUTTAZZO & DAL MASO [10] and CARBONE & SBORDONE [15] have proved that the assumptions of uniform boundedness of the sequences (a_n) and (a_n^{-1}) can be replaced by that of equi-integrability and uniform boundedness of (a_n) in $L^1(\Omega)$. However, outside these two situations the limit functional may dramatically differ from the initial ones. During the last twenty years, many authors carried out diverse asymptotic analyses of functionals of the type (1.1)-(1.2). It appeared that, despite the strong local nature of (1.1)-(1.2), non-classical behaviors such as *killing terms* [28] and *non-local terms* [1, 4, 6, 14, 27] may arise in the limit problem. The first relevant examples of such non-classical phenomena were obtained by FENCHENKO & KHRUSLOV [21] and KHRUSLOV [23]. Using the Deny-Beurling [5, 22] representation of regular Dirichlet forms Mosco [27] derived a general representation for the limit of sequences of the type (1.1)-(1.2). Non-local behaviors were also obtained by BELLIEUD & BOUCHITTÉ [4] in the framework of nonlinear functionals. Proceeding further BRIANE [7] found a condition on the sequence which if satisfied ensures classical behavior and if not then there can appear non-local phenomena. Additionally he established the optimality of this condition in the case of fiber-reinforced media. A subsequent work BRIANE [9] extended the results of [4] and [21] introducing a new approach for the homogenization of periodic media reinforced by aligned high-conductivity fibers. This approach is based on the asymptotic behavior of an auxiliary local spectral problem which measures the non-locality occurring within the reinforced media.

It is clear from these works that high or low conductivity fibers are a good tool for creating non-local phenomena. Recently CHEREDNICHENKO, SMYSHLYAEV and ZHIKOV [16] obtained convolution-type spatial non-local effects from the homogenization of non-uniformly elliptic operators. Using a combination of the two-scale convergence method ([1], [34]) and the classical double-scale asymptotic expansion method, they showed that for arrays of cylinders with the cylinders having very poor conductivity in the transverse direction the macroscopic current and electric fields are linked by a constitutive law which is non-local in the direction parallel to the cylinders axes (see also the earlier papers of SANDRAKOV [31, 32]).

Using a construction based on high conductivity fibers CAMAR-EDDINE & SEPPECHER [13] gave a complete characterization of the closure of the set of functionals of the type (1.1)-(1.2). Their result states that any Dirichlet form on $L^2(\Omega)$ is the Mosco-limit of some sequence of diffusion functionals.

Unusual behaviors such as non-local effects have also been observed [4], [19, 20] in the linear elasticity setting. PIDERI & SEPPECHER [29] gave an example in which the limit of a sequence of isotropic elastic energies

$$F_n(u) = \int_{\Omega} (\alpha_n(x) \|\mathbf{e}(\mathbf{u})\|^2 + \beta_n(x) (\text{Tr}(\mathbf{e}(\mathbf{u})))^2) dx, \quad (1.4)$$

includes some second derivatives of the displacement u . In (1.4), α_n and β_n are the Lamé coefficients of the material, $\mathbf{e}(\mathbf{u})$ represents the strain tensor *i.e.*, the symmetric part of the gradient of u , $\|\mathbf{e}(\mathbf{u})\|$ denotes the Euclidean norm of the matrix $\mathbf{e}(\mathbf{u})$ and $\text{Tr}(\mathbf{e}(\mathbf{u}))$ its trace.

In a recent paper [12] CAMAR-EDDINE and SEPPECHER completely characterized the set of all possible limits of sequences of the type (1.4). They proved that this set coincides with that of all non-negative lower-semicontinuous quadratic functionals which are objective in the sense that they vanish for rigid motions. Another setting where non-local effects have been observed is that of Stokes equations. BRIANE [8] obtained a non-local Brinkman's law from the homogenization of the Stokes equations in a vertical open cylinder with high-contrast viscosity.

Our ultimate goal, using similar approach, is to determine the closure of the set of *isotropic thermoelectric* functionals involving couplings between the temperature and the electric field:

$$\begin{aligned} \int_{\Omega} (\sigma(x)|\nabla u(x)|^2 + \kappa(x)|\nabla v(x)|^2 + 2\alpha(x)\nabla u(x) \cdot \nabla v(x)) dx \\ = \int_{\Omega} \begin{pmatrix} \nabla u(x) \\ \nabla v(x) \end{pmatrix} \cdot \mathbf{L}(x) \begin{pmatrix} \nabla u(x) \\ \nabla v(x) \end{pmatrix}, \end{aligned} \quad (1.5)$$

where σ , κ and α are the elements of the positive definite symmetric thermo-electric matrix

$$\mathbf{L}(x) = \begin{pmatrix} \sigma(x) & \alpha(x) \\ \alpha(x) & \kappa(x) \end{pmatrix} \quad (1.6)$$

of the material occupying the domain Ω . The positive definiteness of $\mathbf{L}(x)$ is equivalent to the condition that

$$\sigma(x) > 0, \quad \kappa(x) > 0 \quad \text{and} \quad \sigma(x)\kappa(x) - \alpha^2(x) > 0 \quad \text{a.e. in } \Omega. \quad (1.7)$$

For the physical interpretation of these coefficients and the potentials u and v , see for example CALLEN [11] or section 2.4 of MILTON [25].

This paper is a modest first step in that direction. We show that a large class of non-local energies of the type

$$\int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \boldsymbol{\mu}(dx, dy) \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \quad (1.8)$$

can be obtained as limits of sequences of functionals of the type (1.5) where $\boldsymbol{\mu}(dx, dy)$ is a positive definite symmetric matrix-valued measure satisfying certain conditions which ensure the continuity of the functional (1.8) in the strong topology of $L^2(\Omega, \mathbb{R}^2)$. The foundation of our argument is provided by the work of CAMAR-EDDINE and SEPPECHER [13].

The structure of the paper is the following: In Section 2 we set up our notations and state our main result. Section 3 introduces some preliminary results (Theorem 2 and Theorem 3) which we use later. Aside from some algebraic lemmas, we will draw upon the method developed in [13] generalizing the key ideas of that approach to some particular functionals depending on two potentials. Note that although the present paper deals with energy functionals acting on two potentials, our result is easily generalized to energies depending on more than two potentials. Moreover, since the results of [13] are proved when the dimension of the physical space is greater than or equal

to three, so is our result. As a matter of fact, the topology of \mathbb{R}^2 seemingly does not allow the construction of a structure connecting two distant points without significantly perturbing the conduction in the remaining part of the medium. To our knowledge the question of the characterization of the closure of conductivity functionals when the dimension of the ambient space is two is still open.

The last section (Section 4) is devoted to the proof of the main result (Theorem 1).

2 Main result

2.1 Notations and definitions

The domain Ω is a bounded open subset of \mathbb{R}^3 . For the sake of simplicity Ω is assumed to be the unit cube $(0, 1)^3$ of \mathbb{R}^3 . The Lebesgue space $L^2(\Omega, \mathbb{R}^2)$ is endowed with the norm $\|u\|_{L^2(\Omega)} := (\int_{\Omega} |u(x)|^2 dx)^{1/2}$. We denote by $H^1(\Omega, \mathbb{R}^2)$ the usual Sobolev space, endowed with its standard norm $\|u\|_{H^1(\Omega, \mathbb{R}^2)} := (\int_{\Omega} |u(x)|^2 dx + \int_{\Omega} |\nabla u(x)|^2 dx)^{1/2}$. We will also denote by $L^\infty(\Omega)$ the set of all essentially bounded Lebesgue measurable functions endowed with the usual norm $\|u\|_{L^\infty(\Omega)} := \inf\{\lambda; |u(x)| \leq \lambda \text{ for a.e. } x \text{ in } \Omega\}$ and by $L_{++}^\infty(\Omega)$ the subset

$$L_{++}^\infty(\Omega) := \{\varphi \in L^\infty(\Omega, \mathbb{R}^+), \varphi^{-1} \in L^\infty(\Omega, \mathbb{R}^+)\}.$$

This space is nothing but the set of all non-degenerate thermal and electrical conductivities.

By $|\mathcal{O}|$ we denote the Lebesgue measure of any Borel subset $\mathcal{O} \subset \otimes$ and by $\int_{\mathcal{O}} u := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} u dx$ we denote the mean value of any function $u \in L^1(\mathcal{O})$.

Let $\mu^{11}(dx, dy)$, $\mu^{12}(dx, dy)$ and $\mu^{22}(dx, dy)$ be three Radon measures on $\Omega \times \Omega$. With the symmetric matrix-valued measure

$$\boldsymbol{\mu}(dx, dy) = \begin{pmatrix} \mu^{11}(dx, dy) & \mu^{12}(dx, dy) \\ \mu^{12}(dx, dy) & \mu^{22}(dx, dy) \end{pmatrix} \quad (2.1)$$

we associate the quadratic non-local energy functional J_μ defined, for any $(u, v) \in L^2(\Omega, \mathbb{R}^2)$, by

$$J_\mu(u, v) := \int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \boldsymbol{\mu}(dx, dy) \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix}. \quad (2.2)$$

Note that in (2.2), the part of the measure $\boldsymbol{\mu}(dx, dy)$ supported by the diagonal $\Delta := \{(x, x), x \in \Omega\}$ does not play any role. We then assume, without loss of generality, that the measures $\mu^{11}(dx, dy)$, $\mu^{12}(dx, dy)$, $\mu^{22}(dx, dy)$ do not charge the diagonal Δ , *i.e.*, they do not give any weight to the diagonal:

$$\mu^{11}(\Delta) = \mu^{12}(\Delta) = \mu^{22}(\Delta) = 0. \quad (2.3)$$

Note also that only the set-symmetric part $(\mu^{kl})^{\text{sym}}(dx, dy)$ of the measure $\mu^{kl}(dx, dy)$, defined for any Borel sets $A \subset \Omega$ and $B \subset \Omega$, by

$$(\mu^{kl})^{\text{sym}}(A \times B) = \frac{1}{2}(\mu^{kl}(A \times B) + \mu^{kl}(B \times A)) \quad (2.4)$$

plays a role in (2.2). This allows us to assume, without loss of generality, that the measures $\mu^{kl}(dx, dy)$, $k, l = 1, 2$, are set-symmetric in the sense that for any Borel sets $A \subset \Omega$ and $B \subset \Omega$ one has:

$$\mu^{kl}(A \times B) = \mu^{kl}(B \times A). \quad (2.5)$$

2.2 Thermoelectric functionals

We denote by \mathfrak{M} the set of *isotropic thermoelectric functionals* with bounded coefficients $\sigma(x)$, $\kappa(x)$ and $\alpha(x)$ satisfying (1.7). Specifically we set

$$\mathfrak{M} := \{J_{\sigma, \kappa, \alpha} ; (\sigma, \kappa, \sigma\kappa - \alpha^2) \in (L_{++}^\infty(\Omega))^3\}, \quad (2.6)$$

where

$$J_{\sigma, \kappa, \alpha}(u, v) := \begin{cases} \int_{\Omega} (\sigma(x)|\nabla u(x)|^2 + \kappa(x)|\nabla v(x)|^2 + 2\alpha(x)\nabla u(x) \cdot \nabla v(x)) dx & \text{if } (u, v) \in H^1(\Omega, \mathbb{R}^2), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.7)$$

Note that there is no uniform bound for σ, κ and α in the Definition (2.6) of \mathfrak{M} . That means: very low or very high coefficients are admissible. It should also be noted that the coupling coefficient α is not necessarily positive. Strictly speaking the functional $J_{\sigma, \kappa, \alpha}$ defined by (2.7) is not the physical energy, but we will refer to it as the energy because it is the energy in the mathematically equivalent magnetoelectric problem (see for instance Milton [25]).

A useful property of the set \mathfrak{M} is its stability under the composition on the right with non-singular linear operators of $L^2(\Omega, \mathbb{R}^2)$. Indeed we have the following:

Property 1 *Let $J_{\sigma, \kappa, \alpha}$ be an element of \mathfrak{M} . Let \mathbf{P} be a 2×2 real and non-singular matrix. Then the composite functional $J_{\sigma, \kappa, \alpha} \circ \mathbf{P}$ defined, for any (u, v) in $L^2(\Omega, \mathbb{R}^2)$, by*

$$(J_{\sigma, \kappa, \alpha} \circ \mathbf{P})(u, v) := J_{\sigma, \kappa, \alpha}(\mathbf{P}(u, v)) \quad (2.8)$$

belongs to \mathfrak{M} .

Proof: Let p_{11} , p_{12} , p_{21} and p_{22} be the elements of the matrix \mathbf{P} . For any (u, v) in $L^2(\Omega, \mathbb{R}^2)$ we have

$$\begin{aligned} (J_{\sigma, \kappa, \alpha} \circ \mathbf{P})(u, v) &= J_{\sigma, \kappa, \alpha}(p_{11}u + p_{12}v, p_{21}u + p_{22}v) \\ &=: J_{\tilde{\sigma}, \tilde{\kappa}, \tilde{\alpha}}(u, v), \end{aligned}$$

where

$$\tilde{\sigma} := p_{11}^2\sigma + p_{21}^2\kappa + 2p_{11}p_{21}\alpha, \quad \tilde{\kappa} := p_{12}^2\sigma + p_{22}^2\kappa + 2p_{12}p_{22}\alpha$$

and

$$\tilde{\alpha} := p_{11}p_{12}\sigma + p_{21}p_{22}\kappa + (p_{11}p_{22} + p_{12}p_{21})\alpha.$$

An elementary computation shows that

$$\begin{aligned}\tilde{\sigma} &= \sigma \left[\left(p_{11} + \frac{\alpha}{\sigma} p_{21} \right)^2 + \left(\frac{p_{21}}{\sigma} \right)^2 (\sigma\kappa - \alpha^2) \right], \\ \tilde{\kappa} &= \sigma \left[\left(p_{12} + \frac{\alpha}{\sigma} p_{22} \right)^2 + \left(\frac{p_{22}}{\sigma} \right)^2 (\sigma\kappa - \alpha^2) \right]\end{aligned}$$

and that

$$\tilde{\sigma}\tilde{\kappa} - \tilde{\alpha}^2 = [\det(\mathbf{P})]^2(\sigma\kappa - \alpha^2),$$

where $\det(\mathbf{P})$ stands for the determinant of the matrix \mathbf{P} . It follows from the positivity of $(\sigma\kappa - \alpha^2)$ and from the non-singularity of the matrix \mathbf{P} that

$$\tilde{\sigma}(x) > 0, \quad \tilde{\kappa}(x) > 0 \quad \text{and} \quad \tilde{\sigma}(x)\tilde{\kappa}(x) - \tilde{\alpha}(x)^2 > 0 \quad \text{a.e. in } \Omega.$$

Therefore the triplet $(\tilde{\sigma}, \tilde{\kappa}, \tilde{\sigma}\tilde{\kappa} - \tilde{\alpha}^2)$ belongs to $(L_{++}^\infty(\Omega))^3$ which implies that the functional $J_{\tilde{\sigma}, \tilde{\kappa}, \tilde{\alpha}}$ belongs to \mathfrak{M} . Property 1 is then proved. \square

2.3 Convergence of functionals

All the functionals considered in this paper are defined on the Lebesgue space $L^2(\Omega, \mathbb{R}^2)$. They are proper, convex and lower-semicontinuous in $L^2(\Omega, \mathbb{R}^2)$. We denote by \mathcal{Q} this set of functionals. The notion of convergence we use is a variational convergence particularly adapted to our problem. We recall its definition and main properties. For more details on this convergence, we refer to [12, 13] and the references therein.

Definition 1 *We say that a sequence of functionals (F_n) in \mathcal{Q} τ -converges to a functional F , and we write $F_n \xrightarrow{\tau} F$, if and only if it satisfies the following three properties:*

i) *Lower-bound inequality: For any sequence (u_n, v_n) converging weakly to some (u, v) in $L^2(\Omega, \mathbb{R}^2)$,*

$$\liminf_{n \rightarrow \infty} F_n(u_n, v_n) \geq F(u, v) . \quad (2.9)$$

ii) *First upper-bound inequality: For every (u, v) in $L^2(\Omega, \mathbb{R}^2)$, there exists an approximating sequence (u_n, v_n) converging to (u, v) strongly in $L^2(\Omega, \mathbb{R}^2)$ such that*

$$\limsup_{n \rightarrow \infty} F_n(u_n, v_n) \leq F(u, v) . \quad (2.10)$$

iii) *Second upper-bound inequality: For any (u, v) in $H^1(\Omega, \mathbb{R}^2)$, there exists a sequence (u_n, v_n) converging to (u, v) in the strong topology of $H^1(\Omega, \mathbb{R}^2)$ such that*

$$\limsup_{n \rightarrow \infty} F_n(u_n, v_n) \leq F(u, v) . \quad (2.11)$$

Remark 1 *Any result that a subset is τ -dense implies the density of that subset for Mosco-convergence in $L^2(\Omega, \mathbb{R}^2)$ and for Γ -convergence in the strong topology of $L^2(\Omega, \mathbb{R}^2)$.*

The notion of τ -convergence is closely related to that of Γ -convergence introduced by De Giorgi [18], [17] which is adapted to study the limit of variational problems and to the notion of Mosco-convergence introduced by U. Mosco [26],

[2] which is more particularly adapted to convex cases. Remark 1 is due to the fact that, the stronger is the notion of convergence, the stronger is a density result.

It is usual to shorten proofs by considering only sequences with bounded energy. Indeed:

Remark 2 *It is clear that a τ -convergence result is proved if for every subsequence (not relabeled) of (F_n) one considers in (i) only sequences (u_n, v_n) with bounded “energy” (i.e., such that $F_n(u_n, v_n) < M < +\infty$) and in (ii) and (iii) only elements (u, v) such that $F(u, v) < +\infty$. In the same way it is usual to shorten the proofs of convergence by considering only sufficiently smooth functions u and v in points (ii) and (iii) of Definition 1: if F is continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, both points (ii) and (iii) of Definition 1 will be proved once proved that*

iv) for any (u, v) in $C^\infty(\Omega, \mathbb{R}^2)$, there exists a sequence (u_n, v_n) converging to (u, v) in the strong topology of $H^1(\Omega, \mathbb{R}^2)$ such that

$$\limsup_{n \rightarrow \infty} F_n(u_n, v_n) \leq F(u, v) . \quad (2.12)$$

The next key property will bring to the fore the fact that τ -convergence is stable when adding elements of a wide class of perturbations.

Property 2 *Let us denote by \mathcal{R} the set of all functionals in \mathcal{Q} which are either continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, or continuous in the strong topology of $H^1(\Omega, \mathbb{R}^2)$ with domain contained in $H^1(\Omega, \mathbb{R}^2)$. We have*

$$(F_n \xrightarrow{\tau} F \quad \text{and} \quad G \in \mathcal{R}) \quad \Rightarrow \quad F_n + G \xrightarrow{\tau} F + G. \quad (2.13)$$

Proof: Let (u_n, v_n) be a sequence weakly converging to some (u, v) in $L^2(\Omega, \mathbb{R}^3)$. As $G \in \mathcal{Q}$ is lower-semicontinuous in the weak topology of $L^2(\Omega, \mathbb{R}^2)$, the τ -convergence of (F_n) to F implies the lower-bound inequality:

$$\liminf_{n \rightarrow \infty} (F_n + G)(u_n, v_n) \geq \liminf_{n \rightarrow \infty} F_n(u_n, v_n) + \liminf_{n \rightarrow \infty} G(u_n, v_n) \geq F(u, v) + G(u, v),$$

and point (i) of Definition 1 is proved. Now, let $(u, v) \in H^1(\Omega, \mathbb{R}^2)$. The τ -convergence of (F_n) to F assures the existence of a sequence (u_n, v_n) converging strongly to (u, v) in $H^1(\Omega, \mathbb{R}^2)$ and satisfying

$$\limsup_{n \rightarrow \infty} F_n(u_n, v_n) \leq F(u, v) . \quad (2.14)$$

As G , in any case, is continuous in the strong topology of $H^1(\Omega, \mathbb{R}^2)$, we have also

$$\limsup_{n \rightarrow \infty} (F_n + G)(u_n, v_n) \leq (F + G)(u, v) \quad (2.15)$$

and point (iii) is proved. Clearly, this proves also point (ii) when the domain of G is contained in $H^1(\Omega, \mathbb{R}^2)$. In the other case, G is continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$. The τ -convergence of (F_n) to F states the existence of a sequence (u_n, v_n) which converges strongly to (u, v) in $L^2(\Omega, \mathbb{R}^2)$ and satisfies the upper-bound (2.14). Due to the continuity of G , inequality (2.15) is satisfied and therefore point (ii) still holds. This completes the proof of Property 2. \square

Definition 2 Let \mathcal{U} be a subset of \mathcal{Q} . We call $\overline{\mathcal{U}}$ the closure of \mathcal{U} and we define it to be the set of all τ -limits of sequences in \mathcal{U} .

The next key property is essential in our proofs.

Property 3 For any subset \mathcal{U} of \mathcal{Q} , we have $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.

The point here is that, like Mosco-convergence and unlike Γ -convergence, the τ -convergence is metrizable at least on a large part of \mathcal{Q} , namely on the set of all functionals in \mathcal{Q} whose domain intersects $H^1(\Omega, \mathbb{R}^2)$.

Property 4 Let \mathcal{U} be a convex cone contained in \mathcal{R} . Let F and G be in $\overline{\mathcal{U}}$ and assume that F belongs to \mathcal{R} . Then $F + G$ belongs to $\overline{\mathcal{U}}$.

Proof: Let (F_n) and (G_n) be two sequences in \mathcal{U} which τ -converge respectively to F and G . Then Property 2 states that $F + G_n$ τ -converges to $F + G$ and that, for any fixed n , $F_m + G_n$ τ -converges to $F + G_n$ as m goes to infinity. We conclude that $F + G$ belongs to $\overline{\mathcal{U}}$ by invoking Property 3.

One consequence of Property 4 is that the closure $\overline{\mathfrak{M}}$ is stable under addition.

In the proof of Theorem 2, we consider the notion of τ -convergence for a sequence of functionals defined on $L^2(\Omega)$. We say that a sequence of functionals $S_n : L^2(\Omega) \rightarrow [0, \infty]$ τ -converges to $S : L^2(\Omega) \rightarrow [0, \infty]$ in $L^2(\Omega)$ if and only if the three properties of Definition 1 are satisfied when the Lebesgue space $L^2(\Omega, \mathbb{R}^2)$ is replaced by $L^2(\Omega)$ and the Sobolev space $H^1(\Omega, \mathbb{R}^2)$ is replaced by $H^1(\Omega)$.

2.4 Statement of the main result

Our main result states that any continuous non-local energy functional of the type (2.2) is the τ -limit of some sequence of thermoelectric energies. Specifically we have the following:

Theorem 1 Let $\mu(dx, dy)$ be a positive definite symmetric matrix-valued measure such that the corresponding non-local energy functional J_μ defined by (2.2) is continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$. Then J_μ belongs to the τ -closure $\overline{\mathfrak{M}}$ of the set of isotropic thermoelectric functionals.

The proof of Theorem 1 is based on three intermediate results we are going to state and prove in the next section.

The first result (Theorem 2) states that for any matrix-valued measure

$$\mu(dx, dy) := \mathbf{A}\nu(dx, dy) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \nu(dx, dy) \quad (2.16)$$

where the constant and symmetric matrix \mathbf{A} is positive definite and where $\nu(dx, dy)$ is a Radon measure on $\Omega \times \Omega$ whose projection $\nu(dx, \Omega)$ on Ω , does not charge polar sets (which are defined as sets with vanishing H^1 -capacity [24]), the corresponding functional J_μ given by (2.2) belongs to the τ -closure $\overline{\mathfrak{M}}$ of the set of isotropic thermoelectric functionals.

The second result (Remark 3) states that any finite sum of measures of the kind (2.16) is realizable in the sense that it belongs to $\overline{\mathfrak{M}}$. This follows from Property 4 and the fact that \mathfrak{M} is a convex cone contained in \mathcal{R} .

The third result (Theorem 3) is a discretization result: it states that if $\mu(dx, dy)$ is a positive definite symmetric matrix-valued measure of the type (2.1) on $\Omega \times \Omega$ which makes the corresponding energy functional J_μ continuous in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, then J_μ is the τ -limit of a sequence of functionals associated to a sequence of finite sums of measures of the type (2.16). This result is obtained by a discretization procedure introduced in [13].

3 Preliminary results

We begin by proving the result of Theorem 1 for a particular family of measures $\mu(dx, dy)$; namely for those measures with elements all proportional to the same measure $\nu(dx, dy)$.

Theorem 2 *Let $\mu(dx, dy)$ be as in the paragraph containing (2.16). Then, the corresponding non-local energy functional J_μ given by (2.2) belongs to the τ -closure $\overline{\mathfrak{M}}$ of the set of isotropic thermoelectric functionals.*

Before proceeding to the proof of Theorem 2 let us establish two lemmas we use in the proof.

3.1 Two useful lemmas

Lemma 1 *Let F_n and $G_n : L^2(\Omega) \rightarrow [0, +\infty]$ be two sequences τ -converging, in $L^2(\Omega)$, to F and G , respectively. Then, the sequence of functionals $J_n(u, v) := F_n(u) + G_n(v)$ τ -converges to the functional $J(u, v) := F(u) + G(v)$.*

Proof of Lemma 1: The point here is that the functionals F_n and G_n act on two independent variables u and v . The result is proved by two inequalities.

Lower-bound inequality: Let (u_n, v_n) be a sequence converging to some (u, v) weakly in $L^2(\Omega, \mathbb{R}^2)$ with bounded energy *i.e.*, such that $J_n(u_n, v_n) \leq M < \infty$. Then, the τ -convergence in $L^2(\Omega)$ of F_n and G_n to F and G , respectively, leads to

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_n(u_n, v_n) &= \liminf_{n \rightarrow \infty} (F_n(u_n) + G_n(v_n)) \\ &\geq \liminf_{n \rightarrow \infty} F_n(u_n) + \liminf_{n \rightarrow \infty} G_n(v_n) \\ &\geq F(u) + G(v) \\ &= J(u, v). \end{aligned} \tag{3.1}$$

Upper-bound inequality: Let $(u, v) \in L^2(\Omega, \mathbb{R}^2)$ such that $J(u, v) < \infty$. Again, the τ -convergence in $L^2(\Omega)$ of F_n and G_n to F and G , respectively, assures the existence of two independent approximating sequences (u_n) and (v_n) strongly converging in $L^2(\Omega)$ to u and v , respectively, such that $\limsup F_n(u_n) \leq F(u)$ and $\limsup G_n(v_n) \leq G(v)$. It follows that the sequence (u_n, v_n) converges to

(u, v) strongly in $L^2(\Omega, \mathbb{R}^2)$ and satisfies

$$\begin{aligned}
\limsup_{n \rightarrow \infty} J_n(u_n, v_n) &= \limsup_{n \rightarrow \infty} (F_n(u_n) + G_n(v_n)) \\
&\leq \limsup_{n \rightarrow \infty} F_n(u_n) + \limsup_{n \rightarrow \infty} G_n(v_n) \\
&\leq F(u) + G(v) \\
&= J(u, v).
\end{aligned} \tag{3.2}$$

Lemma 1 is then proved. \square

Lemma 2 *Let \mathbf{P} be a 2×2 real non-singular matrix. Let (J_n) be a sequence τ -converging to some J . Then the sequence (\tilde{J}_n) , defined on $L^2(\Omega, \mathbb{R}^2)$ by $\tilde{J}_n(u, v) := J_n(\mathbf{P}(u, v))$, τ -converges to \tilde{J} , defined on $L^2(\Omega, \mathbb{R}^2)$ by $\tilde{J}(u, v) := J(\mathbf{P}(u, v))$.*

The proof of Lemma 2 is straightforward. Indeed, let (u_n, v_n) be a sequence weakly converging in $L^2(\Omega, \mathbb{R}^2)$ to some (u, v) such that $\tilde{J}_n(u_n, v_n) \leq M < \infty$. By linearity the sequence $(\mathbf{P}(u_n, v_n))$ converges weakly in $L^2(\Omega, \mathbb{R}^2)$ to $(\mathbf{P}(u, v))$. From the τ -convergence of (J_n) to J it follows that

$$\liminf_{n \rightarrow \infty} \tilde{J}_n(u_n, v_n) = \liminf_{n \rightarrow \infty} J_n(\mathbf{P}(u_n, v_n)) \geq J(\mathbf{P}(u, v)) = \tilde{J}(u, v). \tag{3.3}$$

Now let $(u, v) \in L^2(\Omega, \mathbb{R}^2)$ such that $\tilde{J}(u, v) < \infty$. By the τ -convergence of (J_n) to J , there exists an approximating sequence (ξ_n, η_n) strongly converging to $\mathbf{P}(u, v)$ in $L^2(\Omega, \mathbb{R}^2)$ such that

$$\limsup_{n \rightarrow \infty} J_n(\xi_n, \eta_n) \leq J(\mathbf{P}(u, v)). \tag{3.4}$$

Taking (3.4) into account and setting $(u_n, v_n) := \mathbf{P}^{-1}(\xi_n, \eta_n)$ (where \mathbf{P}^{-1} stands for the inverse matrix of \mathbf{P}) provides us with a sequence (u_n, v_n) converging to (u, v) strongly in $L^2(\Omega, \mathbb{R}^2)$, and satisfying

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \tilde{J}_n(u_n, v_n) &= \limsup_{n \rightarrow \infty} J_n(\mathbf{P}(u_n, v_n)) \\
&= \limsup_{n \rightarrow \infty} J_n(\xi_n, \eta_n) \\
&\leq J(\mathbf{P}(u, v)) = \tilde{J}(u, v).
\end{aligned} \tag{3.5}$$

Inequality (3.5) together with (3.3) proves the τ -convergence of (\tilde{J}_n) to \tilde{J} . This concludes the proof of Lemma 2. \square

Now that the two lemmas are proved, let us proceed with the proof of Theorem 2.

3.2 Proof of Theorem 2

The eigenvalues λ_1 and λ_2 of the positive definite symmetric matrix

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$$

are positive. Let \mathbf{a} and \mathbf{b} be the two eigenvectors of \mathbf{A} having norms, $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, respectively. Then, the matrix \mathbf{A} takes the form $\mathbf{A} = \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}$. The associated energy

$$J_\mu(u, v) = \int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \boldsymbol{\mu}(dx, dy) \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \quad (3.6)$$

can be written in the form

$$J_\mu(u, v) = F(\mathbf{P}(u, v)),$$

with

$$\mathbf{P} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

and

$$\begin{aligned} F(u, v) &= \int_{\Omega \times \Omega} (u(x) - u(y))^2 \nu(dx, dy) + \int_{\Omega \times \Omega} (v(x) - v(y))^2 \nu(dx, dy) \\ &=: S(u) + Q(v) \quad \text{for any } (u, v) \in L^2(\Omega, \mathbb{R}^2). \end{aligned} \quad (3.7)$$

Therefore, owing to Lemma 2 and Property 1, it is sufficient to prove that the functional F , defined on $L^2(\Omega, \mathbb{R}^2)$ by (3.7), belongs to the closure $\overline{\mathfrak{M}}$ of the set of thermoelectric functionals. On the other hand, by Theorem 1 of [13], there are two sequences (S_n) and (Q_n) defined on $L^2(\Omega)$ by

$$S_n(u) := \begin{cases} \int_{\Omega} \sigma_n(x) |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \quad (3.8)$$

and

$$Q_n(v) := \begin{cases} \int_{\Omega} \kappa_n(x) |\nabla v(x)|^2 dx & \text{if } v \in H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \quad (3.9)$$

where $\sigma_n \in L_{++}^\infty(\Omega)$ and $\kappa_n \in L_{++}^\infty(\Omega)$, such that (S_n) τ -converges in $L^2(\Omega)$ to S and (Q_n) to Q . Lemma 1 states that the sequence $(J_{\sigma_n, \kappa_n, 0})$ defined on $L^2(\Omega, \mathbb{R}^2)$ by

$$J_{\sigma_n, \kappa_n, 0}(u, v) := S_n(u) + Q_n(v),$$

τ -converges to $F(u, v) = S(u) + Q(v)$. Therefore the functional F belongs to $\overline{\mathfrak{M}}$. We conclude the proof of Theorem 2 by invoking Lemma 2 and Property 1. \square

Since \mathfrak{M} is a convex cone contained in \mathcal{R} , owing to Property 4 we have the following:

Remark 3 *Any finite sum of measures of the kind (2.16) is realizable, in the sense that the associated functional belongs to the τ -closure $\overline{\mathfrak{M}}$ of the set of isotropic thermoelectric functionals.*

3.3 Discretization of a non-local functional

The next result (Theorem 3) is a discretization result. We use the concept of atomic interaction introduced in [13] to prove that a wide class of non-local energy functionals of the type J_μ defined by (2.2) belongs to the τ -closure $\overline{\mathfrak{M}}$ of the set of isotropic thermoelectric functionals. By *elementary interaction* we mean any element of the set

$$\mathcal{E} := \{ \delta_{x+w}(dy) f(x) \mathbf{1}_\Omega(x+w) dx, w \in \Omega_2, f \in L^\infty(\Omega), f \geq 0 \}, \quad (3.10)$$

where $\mathbf{1}_\Omega$ is the characteristic function of the set Ω , and Ω_2 represents the set of vectors in \mathbb{R}^3 with dyadic components $\Omega_2 := \{w \in \mathbb{R}^3, \exists p \in \mathbb{N} : 2^p w \in \mathbb{N}^3\}$. This restriction to vectors with dyadic components is purely technical. By *atomic interaction* we mean any finite combination of elementary interactions. We denote the set of such measures by

$$\mathcal{A} := \left\{ \sum_{i=1}^n \delta_{x+w_i}(dy) f_i(x) \mathbf{1}_\Omega(x+w_i) dx, n \in \mathbb{N}, w_i \in \Omega_2, f_i \in L^\infty(\Omega), f_i \geq 0 \right\}.$$

The sets \mathcal{E} and \mathcal{A} provide examples of scalar measures $\nu(dx, dy)$ whose projections $\nu(dx, \Omega)$ on Ω are absolutely continuous with respect to dx . Therefore, any functional of the kind $\int_{\Omega \times \Omega} (u(x) - u(y))^2 \nu(dx, dy)$ associated with any measure $\nu(dx, dy)$ in \mathcal{E} or \mathcal{A} is strongly continuous in $L^2(\Omega, \mathbb{R}^2)$ (see for instance Section 2.2 of the reference [13]).

Let $\boldsymbol{\mu}(dx, dy)$ be a symmetric matrix-valued measure of the type (2.1) whose projection $\boldsymbol{\mu}(dx, \Omega)$ on Ω is absolutely continuous with respect to the Lebesgue measure dx in the sense that all its components $\mu^{11}(dx, dy)$, $\mu^{12}(dx, dy)$ and $\mu^{22}(dx, dy)$ have projections $\mu^{11}(dx, \Omega)$, $\mu^{12}(dx, \Omega)$ and $\mu^{22}(dx, \Omega)$ which are absolutely continuous with respect to dx . Note that the absolute continuity of $\boldsymbol{\mu}(dx, \Omega)$ with respect to dx is equivalent to the continuity, in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, of the associated functional J_μ defined by (2.2) (see for instance [13]). With this measure $\boldsymbol{\mu}(dx, dy)$ we associate a sequence $(\boldsymbol{\mu}_n)$ of measures as follows: Let n denote a sequence of integers tending to infinity of the form $n = 2^{q_n}$ (with $q_n \in \mathbb{N}$). We divide the domain Ω in n^3 elementary cubes

$$\Omega_I^n := \left(\frac{i-1}{n}, \frac{i}{n} \right) \times \left(\frac{j-1}{n}, \frac{j}{n} \right) \times \left(\frac{k-1}{n}, \frac{k}{n} \right), \quad (3.11)$$

with centers c_I^n where $I := (i, j, k)$ belongs to $\{1, \dots, n\}^3$ which we identify with $\{1, \dots, n^3\}$. For any component $\mu^{ij}(dx, dy)$ of $\boldsymbol{\mu}(dx, dy)$, we define the sequence of components (μ_n^{ij}) by

$$\mu_n^{ij}(dx, dy) := \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} (a_{II'}^n)^{ij} \mathbf{1}_{\Omega_I^n}(x) \delta_{x+w_{II'}^n}(dy) \mathbf{1}_\Omega(x+w_{II'}^n) dx \quad (3.12)$$

where the weight $(a_{II'}^n)^{ij} := n^3 \mu^{ij}(\Omega_I^n \times \Omega_{I'}^n)$ normalizes the measure $\mu^{ij}(dx, dy)$ and where the vector $w_{II'}^n$ is defined by $w_{II'}^n := c_{I'}^n - c_I^n$. For any $n \geq 1$ we define on $L^2(\Omega, \mathbb{R}^2)$ the functional

$$J_{\mu_n}(u, v) = \int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \boldsymbol{\mu}_n(dx, dy) \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix}. \quad (3.13)$$

The aim of this section is to prove the following:

Theorem 3 *The sequence of functionals (J_{μ_n}) defined by (3.12)-(3.13) τ -converges to J_μ .*

Proof: The result is proved by two inequalities.

Lower-bound inequality: Let (u_n, v_n) be a sequence with bounded energy weakly converging to some (u, v) in $L^2(\Omega, \mathbb{R}^2)$. With (u_n, v_n) we associate a new sequence (\bar{u}_n, \bar{v}_n) in $L^2(\Omega, \mathbb{R}^2)$ as follows: For any $n \geq 1$ we define the piecewise constant functions \bar{u}_n and \bar{v}_n by

$$\bar{u}_n(x) := \sum_{I=1}^{n^3} \left(\int_{\Omega_I^n} u_n \right) \mathbf{1}_{\Omega_I^n}(x) \quad \text{and} \quad \bar{v}_n(x) := \sum_{I=1}^{n^3} \left(\int_{\Omega_I^n} v_n \right) \mathbf{1}_{\Omega_I^n}(x). \quad (3.14)$$

It is easy to verify that the sequence (\bar{u}_n, \bar{v}_n) weakly converges in $L^2(\Omega, \mathbb{R}^2)$ to (u, v) . Taking into account the definitions (3.12) and (3.14) we have

$$\begin{aligned} J_{\mu_n}(u_n, v_n) &= \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} n^3 |\Omega_I^n| \int_{\Omega_I^n} \left(\mu^{11}(\Omega_I^n \times \Omega_{I'}^n) [u_n(x) - u_n(x + w_{II'}^n)]^2 \right. \\ &\quad + 2\mu^{12}(\Omega_I^n \times \Omega_{I'}^n) \left(u_n(x) - u_n(x + w_{II'}^n) \right) \cdot \left(v_n(x) - v_n(x + w_{II'}^n) \right) \\ &\quad \left. + \mu^{22}(\Omega_I^n \times \Omega_{I'}^n) [v_n(x) - v_n(x + w_{II'}^n)]^2 \right) dx. \end{aligned}$$

Moreover, for any $n \geq 1$ and any pair $(I, I') \in \{1, \dots, n^3\}^2$ the function f defined on \mathbb{R}^2 by

$$f(r, s) \equiv \begin{pmatrix} r \\ s \end{pmatrix} \cdot \begin{pmatrix} \mu^{11}(\Omega_I^n \times \Omega_{I'}^n) & \mu^{12}(\Omega_I^n \times \Omega_{I'}^n) \\ \mu^{12}(\Omega_I^n \times \Omega_{I'}^n) & \mu^{22}(\Omega_I^n \times \Omega_{I'}^n) \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}, \quad (3.15)$$

where $\mathbf{a} \cdot \mathbf{b}$ denotes the usual scalar product of two vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^2 , is quadratic and positive, therefore convex. Then, due to Jensen's inequality we have

$$\begin{aligned} J_{\mu_n}(u_n, v_n) &\geq \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} n^3 |\Omega_I^n| \left\{ \mu^{11}(\Omega_I^n \times \Omega_{I'}^n) \left(\int_{\Omega_I^n} u_n(x) dx - \int_{\Omega_I^n} u_n(x + w_{II'}^n) dx \right)^2 \right. \\ &\quad + \mu^{22}(\Omega_I^n \times \Omega_{I'}^n) \left(\int_{\Omega_I^n} v_n(x) dx - \int_{\Omega_I^n} v_n(x + w_{II'}^n) dx \right)^2 \\ &\quad + 2\mu^{12}(\Omega_I^n \times \Omega_{I'}^n) \left(\int_{\Omega_I^n} u_n(x) dx - \int_{\Omega_I^n} u_n(x + w_{II'}^n) dx \right) \\ &\quad \left. \times \left(\int_{\Omega_I^n} v_n(x) dx - \int_{\Omega_I^n} v_n(x + w_{II'}^n) dx \right) \right\} \\ &\geq J_{\mu_n}(\bar{u}_n, \bar{v}_n). \end{aligned}$$

As the measures $\mu^{ij}(dx, \Omega)$ are all absolutely continuous with respect to the Lebesgue measure dx , they do not charge any two-dimensional manifold of \mathbb{R}^3 ; in particular, they do not give any weight to the sets $\partial\Omega_I^n$ (i.e., the boundaries of the elementary cubes Ω_I^n). It follows that

$$J_{\mu_n}(\bar{u}_n, \bar{v}_n) = J_\mu(\bar{u}_n, \bar{v}_n).$$

Moreover the functional J_μ is strongly continuous in $L^2(\Omega, \mathbb{R}^2)$. It is then strongly lower-semicontinuous in $L^2(\Omega, \mathbb{R}^2)$ and therefore weakly lower - semi-continuous in $L^2(\Omega, \mathbb{R}^2)$ since convex. Using the fact that the sequence (\bar{u}_n, \bar{v}_n) weakly converges to (u, v) in $L^2(\Omega, \mathbb{R}^2)$, we get

$$\liminf_{n \rightarrow \infty} J_{\mu_n}(u_n, v_n) \geq \liminf_{n \rightarrow \infty} J_\mu(\bar{u}_n, \bar{v}_n) \geq J(u, v). \quad (3.16)$$

Inequality (3.16) yields the lower-bound inequality.

Upper-bound inequality: Let $(u, v) \in L^2(\Omega, \mathbb{R}^2)$ such that $J_\mu(u, v) < \infty$. By a density argument we can assume that u and v belong to $C_0^1(\Omega)$. With (u, v) we associate the sequence $(\tilde{u}_n, \tilde{v}_n)$ defined on $L^2(\Omega, \mathbb{R}^2)$ by

$$\tilde{u}_n(x) := \sum_{I=1}^{n^3} \left(\int_{\Omega_I^n} u \right) \mathbf{1}_{\Omega_I^n}(x) \quad \text{and} \quad \tilde{v}_n(x) := \sum_{I=1}^{n^3} \left(\int_{\Omega_I^n} v \right) \mathbf{1}_{\Omega_I^n}(x). \quad (3.17)$$

Thanks to the regularity of u and v the sequence $(\tilde{u}_n, \tilde{v}_n)$ uniformly converges to (u, v) . Therefore $(\tilde{u}_n, \tilde{v}_n)$ strongly converges to (u, v) in $L^2(\Omega, \mathbb{R}^2)$. We moreover claim that

$$\begin{aligned} J_{\mu_n}(u_n, v_n) &= J_{\mu_n}(\tilde{u}_n, \tilde{v}_n) + O\left(\frac{1}{n}\right) \\ &= J_\mu(\tilde{u}_n, \tilde{v}_n) + O\left(\frac{1}{n}\right). \end{aligned} \quad (3.18)$$

It follows, from (3.18) and from the continuity of the functional J_μ in the strong topology of $L^2(\Omega, \mathbb{R}^2)$, that

$$\limsup_{n \rightarrow \infty} J_{\mu_n}(u_n, v_n) = \limsup_{n \rightarrow \infty} J_\mu(\tilde{u}_n, \tilde{v}_n) = J(u, v).$$

To conclude the proof of the τ -convergence of the sequence (J_{μ_n}) to J_μ it remains to prove the claim (3.18).

Proof of the claim (3.18): Taking into account the definitions (3.12), (3.13) and (3.17) we have

$$\begin{aligned} |J_{\mu_n}(u_n, v_n) - J_{\mu_n}(\tilde{u}_n, \tilde{v}_n)| &\leq \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} \left\{ (a_{II'}^n)^{11} \left| \int_{\Omega_I^n} T_1^n(x) dx \right| + (a_{II'}^n)^{22} \left| \int_{\Omega_I^n} T_2^n(x) dx \right| \right. \\ &\quad \left. + 2(a_{II'}^n)^{12} \left| \int_{\Omega_I^n} T_3^n(x) dx \right| \right\}, \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} T_1^n(x) &:= [u(x) - u(x + w_{II'}^n)]^2 - [\tilde{u}_n(x) - \tilde{u}_n(x + w_{II'}^n)]^2, \\ T_2^n(x) &:= [v(x) - v(x + w_{II'}^n)]^2 - [\tilde{v}_n(x) - \tilde{v}_n(x + w_{II'}^n)]^2 \end{aligned}$$

and

$$\begin{aligned} T_3^n(x) &:= [u(x) - u(x + w_{II'}^n)] [v(x) - v(x + w_{II'}^n)] \\ &\quad - [\tilde{u}_n(x) - \tilde{u}_n(x + w_{II'}^n)] [\tilde{v}_n(x) - \tilde{v}_n(x + w_{II'}^n)]. \end{aligned} \quad (3.20)$$

Let us now estimate the three terms of (3.19) separately. We have, for a.e. $x \in \Omega_I^n$

$$\begin{aligned} |T_1^n(x)| &\leq (|u(x) - \tilde{u}_n(x)| + |\tilde{u}_n(x + w_{II'}^n) - u(x + w_{II'}^n)|) \\ &\quad \times (|u(x)| + |u(x + w_{II'}^n)| + |\tilde{u}_n(x)| + |\tilde{u}_n(x + w_{II'}^n)|) \\ &\leq \left(\left| \int_{\Omega_I^n} [u(x) - u(y)] dy \right| + \left| \int_{\Omega_{II'}^n} [u(y) - u(x + w_{II'}^n)] dx \right| \right) \\ &\quad \times (|u(x)| + |u(x + w_{II'}^n)| + |\tilde{u}_n(x)| + |\tilde{u}_n(x + w_{II'}^n)|). \end{aligned} \quad (3.21)$$

Moreover since the diagonal of Ω_I^n has length $\sqrt{3}/n$ and since u is regular on Ω we have

$$\left| \int_{\Omega_I^n} [u(x) - u(y)] dy \right| \leq \|\nabla u\|_\infty \frac{\sqrt{3}}{n} \quad \text{a.e. in } \Omega_I^n \quad (3.22)$$

and

$$\left| \int_{\Omega_{II'}^n} [u(y) - u(x + w_{II'}^n)] dx \right| \leq \|\nabla u\|_\infty \frac{\sqrt{3}}{n} \quad \text{a.e. in } \Omega_I^n. \quad (3.23)$$

Using the fact that for a.e. $x \in \Omega_I^n$

$$(|u(x)| + |u(x + w_{II'}^n)| + |\tilde{u}_n(x)| + |\tilde{u}_n(x + w_{II'}^n)|) \leq 4\|u\|_\infty,$$

it follows from (3.21)-(3.23) that

$$|T_1^n(x)| \leq (8\sqrt{3}\|\nabla u\|_\infty\|u\|_\infty) \frac{1}{n} \quad \text{a.e. in } \Omega_I^n. \quad (3.24)$$

Similarly we get

$$|T_2^n(x)| \leq (8\sqrt{3}\|\nabla v\|_\infty\|v\|_\infty) \frac{1}{n} \quad \text{a.e. in } \Omega_I^n. \quad (3.25)$$

To estimate $T_3^n(x)$ we first notice that

$$\begin{aligned} T_3^n(x) &= \frac{1}{4} [(a_n(x) - \tilde{a}_n(x) + b_n(x) - \tilde{b}_n(x)) (a_n(x) + \tilde{a}_n(x) + b_n(x) + \tilde{b}_n(x)) \\ &\quad + (\tilde{a}_n(x) - a_n(x) + b_n(x) - \tilde{b}_n(x)) (a_n(x) + \tilde{a}_n(x) - b_n(x) - \tilde{b}_n(x))], \end{aligned} \quad (3.26)$$

where

$$a_n(x) = u(x) - u(x + w_{II'}^n), \quad \tilde{a}_n(x) = \tilde{u}_n(x) - \tilde{u}_n(x + w_{II'}^n)$$

$$b_n(x) = v(x) - v(x + w_{II'}^n) \quad \text{and} \quad \tilde{b}_n(x) = \tilde{v}_n(x) - \tilde{v}_n(x + w_{II'}^n).$$

One easily verifies, using (3.22), (3.23) and the analogous estimates for v that for a.e. $x \in \Omega_I^n$, we have

$$|a_n(x) - \tilde{a}_n(x)| \leq 2\|\nabla u\|_\infty \frac{\sqrt{3}}{n} \quad \text{and} \quad |b_n(x) - \tilde{b}_n(x)| \leq 2\|\nabla v\|_\infty \frac{\sqrt{3}}{n}. \quad (3.27)$$

Moreover we have, for a.e. $x \in \Omega_I^n$

$$|a_n(x) + \tilde{a}_n(x) + b_n(x) + \tilde{b}_n(x)| \leq 4(u\|_\infty + \|v\|_\infty) \quad (3.28)$$

and

$$|a_n(x) + \tilde{a}_n(x) - b_n(x) - \tilde{b}_n(x)| \leq 4(\|u\|_\infty + \|v\|_\infty). \quad (3.29)$$

It follows from (3.26)-(3.29) that

$$|T_3^n(x)| \leq \frac{4\sqrt{3}}{n}(\|\nabla u\|_\infty + \|\nabla v\|_\infty)(\|u\|_\infty + \|v\|_\infty) \quad \text{a.e. in } \Omega_I^n. \quad (3.30)$$

Finally substituting the inequalities (3.24), (3.25) and (3.30) back in (3.19) allows us to establish the claim (3.18). Indeed, using (3.24) we obtain the following estimate for the first term in (3.19):

$$\begin{aligned} \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} (a_{II'}^n)^{11} \left| \int_{\Omega_I^n} T_1^n(x) dx \right| &\leq \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} (a_{II'}^n)^{11} \int_{\Omega_I^n} |T_1^n(x)| dx \\ &\leq (8\sqrt{3}\|\nabla u\|_\infty\|u\|_\infty) \frac{1}{n} \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} (a_{II'}^n)^{11} \int_{\Omega_I^n} dx \\ &\leq (8\sqrt{3}\|\nabla u\|_\infty\|u\|_\infty) \frac{1}{n} \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} (a_{II'}^n)^{11} \frac{1}{n^3} \\ &\leq (8\sqrt{3}\|\nabla u\|_\infty\|u\|_\infty) \frac{1}{n} \sum_{I=1}^{n^3} \sum_{I'=1}^{n^3} \mu^{11}(\Omega_I^n \times \Omega_{I'}^n) \\ &\leq 8\sqrt{3}(\|\nabla u\|_\infty\|u\|_\infty) \mu^{11}(\Omega \times \Omega) \frac{1}{n}. \end{aligned} \quad (3.31)$$

Estimates for the remaining terms in (3.19) are obtained by similar analysis. The claim (3.18) is then proved. This completes the proof of the τ -convergence of the sequence (J_{μ_n}) to J_μ . Therefore Theorem 3 is proved. \square

With the preliminary results proved, we are now in a position to provide the proof of our main result (Theorem 1).

4 Proof of Theorem 1

Let (J_{μ_n}) be the sequence defined by (3.13). For any fixed $n \geq 1$ and any fixed pair $(I, I') \in \{1, \dots, n^3\}^2$, the functional J_{μ_n} is a finite sum of energies of the type

$$\int_{\Omega \times \Omega} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} u(x) - u(y) \\ v(x) - v(y) \end{pmatrix} \nu(dx, dy), \quad (4.1)$$

where (A_{ij}) is a 2×2 constant positive definite symmetric matrix and $\nu(dx, dy)$ is a Radon measure on $\Omega \times \Omega$ whose projection $\nu(dx, \Omega)$ on Ω is absolutely continuous with respect to dx . Owing to Theorem 2, any functional of the type (4.1) belongs to the closure $\overline{\mathfrak{M}}$. Therefore, Remark 3 states that for any fixed

$n \geq 1$ the functional J_{μ_n} belongs to the closure $\overline{\mathfrak{M}}$. Moreover, by Theorem 3 the sequence (J_{μ_n}) τ -converges to J_μ . We conclude the proof of Theorem 1 by invoking Property 3. \square

Acknowledgements

We are grateful to the National Science Foundation for support through the grant DMS-0108626 A001, and to the Army Research Office for support through the grant DAAD19-03-1-0339.

References

- [1] ALLAIRE G., *Homogenization and two-scale convergence*, SIAM J. Math. Anal. Vol. 23, no. 6, pp. 1482-1518 (1992).
- [2] ATTOUCH H., *Variational Convergence for Functions and Operators*, Pitman, London, (1984).
- [3] BAKHVALOV N.S, PANASENKO G.P., *Homogenization: averaging processes in periodic media. Mathematical problems in the mechanics of composite materials*, Translated from the Russian by D. Leites. Mathematics and its Applications (Soviet Series), 36. Kluwer Academic Publishers Group, Dordrecht, 1989.
- [4] BELLIEUD M., BOUCHITTÉ G., *Homogenization of elliptic problems in a fiber reinforced structure. Non local effects*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 26 (1998), pp. 407-436.
- [5] BEURLING A., DENY J., *Dirichlet spaces*, Proc. Nat. Acad. Sci. U.S.A., 45(1959), 208-215.
- [6] BRIANE M., TCHOU N., *Fibered microstructures for some nonlocal Dirichlet forms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), no. 3-4, pp. 681-711.
- [7] BRIANE M., *Homogenization of non-uniformly bounded operators: critical barrier for nonlocal effects*, Arch. Ration. Mech. Anal, 164, no.1, pp. 73-101 (2002).
- [8] BRIANE M., *Homogenization of the Stokes equations with high-contrast viscosity*, J. Math. Pures Appl. (9) 82, no.7, pp. 843-876 (2003).
- [9] BRIANE M., *Homogenization of high-conductivity periodic problems: application to a general distribution of one-directional fibers*, SIAM J. Math. Anal. Vol. 35, no. 1, pp. 33-60 (2003)
- [10] BUTTAZZO G., DAL MASO G., *Γ -limits of integral functionals*, J. Analyse Math., 37, (1980), 145-185.
- [11] CALLEN H. B., *Thermodynamics: An introduction to the physical Theories of Equilibrium Thermostatics and Irreversible Thermodynamics*, New York / London / Sydney, Australia: John Wiley and Sons, (1960), pp. 293-302.

- [12] CAMAR-EDDINE M., SEPPECHER P., *Determination of the closure of the set of elasticity functionals*, Arch. Ration. Mech. Anal, Vol. 170, no. 3 (2003), pp. 211-245.
- [13] CAMAR-EDDINE M., SEPPECHER P., *Closure of the set of diffusion functionals with respect to Mosco-convergence*, Mathematical Models and Methods in Applied Sciences, Vol. 12, No. 8 (2002), pp. 1153-1176.
- [14] CAMAR-EDDINE M., SEPPECHER P., *Non-local interactions resulting from the homogenization of a linear diffusive medium*, C.R. Acad. Sci. Paris, 332, Série I (2001), pp. 485-490.
- [15] CARBONE L., SBORDONE C., *Some properties of Gamma-limits of integral functionals*, Annali di Matematica Pura ed Applicata 122 (1979), pp. 1-60.
- [16] CHEREDNICHENKO K.D, SMYSHLYAEV V.P., ZHIKOV V.V, *Non-local homogenized limits for periodic composite media with a highly anisotropic component*, To be submitted to the Proceedings of the Royal Society of London A.
- [17] DAL MASO G., *An introduction to Γ -convergence*, Progress in linear diff. eq. and their app., Birkhäuser, Boston, (1993).
- [18] DE GIORGI E., *Su un tipo di convergenza variazionale*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 58 (1975), pp. 842-850.
- [19] DRUGAN W. J., WILLIS J. R., *A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites*, J. Mech. Phys. Solids, Vol. 44, no. 4, pp. 497-524 (1996).
- [20] DRUGAN W. J., *Micromechanics-based variational estimates for higher-order nonlocal constitutive equation and optimal choice of effective moduli for elastic composites*, J. Mech. Phys. Solids, Vol. 48, no. 6-7, pp. 1359-1387 (2000).
- [21] FENCHENKO V.N., KHRUSLOV E. YA, *Asymptotic behavior of solutions of differential equations with a strongly oscillating coefficient matrix that does not satisfy a uniform boundedness condition*, Dokl. Akad. Nauk Ukrain. SSR Ser. A, no. 4, pp. 24-27 (1981).
- [22] FUKUSHIMA M., *Dirichlet Forms and Markov Processes*, North Holland (1980).
- [23] KHRUSLOV E. YA, *Homogenized models of composite media*, Composite Media and Homogenization Theory, G. Dal Maso and G. F. Dell' Antonio editors, in Progress in Nonlinear Differential Equations and their Applications, Birkhäuser pp. 159-182 (1991).
- [24] MALÝ J., ZIEMER W. P., *Fine regularity of solutions of elliptic partial differential equations*, American Mathematical Society, Providence, RI, (1997).
- [25] MILTON G.W, *The theory of composites*, Cambridge University Press, Cambridge, U.K (2002).

- [26] MOSCO U., *Approximation of the solutions of some variational inequalities*, Ann. Scuola Normale Sup. Pisa, 21, pp. 373-394 (1967).
- [27] MOSCO U., *Composite media and asymptotic Dirichlet forms*, Journal of Functional Analysis, 123, 368-421 (1994).
- [28] MURAT F., *A strange term brought from somewhere else*, Nonlinear differential equations and their applications. Collège de France Seminar, Vol II, pp. 98 -138, 389-390, (Paris, 1979/1980), Res.Notes in Math., 60, Pitman, Boston, Mass.-London, (1982).
- [29] PIDERI C., SEPPECHER P., *A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium*, Continuum Mech.and Thermodyn., 9, pp. 241-257,(1997).
- [30] SMYSHLYAEV V.P., CHEREDNICHENKO K.D, *On rigorous derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media*, The J. R. Willis 60th anniversary volume. J. Mech. Phys. Solids, Vol. 48, no. 6-7, pp. 1325-1357 (2000).
- [31] SANDRAKOV G.V., *Averaging of non-stationary equations with contrast coefficients*. (Russian) Dokl. Akad. Nauk 355, no. 5, pp. 605-608 (1997).
- [32] SANDRAKOV G.V., *Averaging of non-stationary problems in the theory of strongly nonhomogeneous elastic media*. (Russian) Dokl. Akad. Nauk 358, no. 3, pp. 308-311 (1998).
- [33] TARTAR,L., *Estimations de coefficients homogénéisés*, Computing methods in applied sciences and engineering, R. Glowinski, J.L. Lions eds, Lecture Notes in Math., 704, Springer Verlag (1978), pp. 364-373.
- [34] ZHIKOV V.V., *On an extension of the method of two-scale convergence and its applications*, Sbornik. Math. Vol. 191 (7-8), pp. 973-1014 (1992).